KTH Teknikvetenskap

SF2723 Topics in Mathematics - Matrix groups
Homework Assignment 7
2008-10-15

The solutions should be handed in no later than October 29, 2008. The final grade will be based upon the total score on the homework and on the oral exam. The total maximal score on the homework assignments is 200 and in order to pass, at least 100 is required.

In order to get full score on each problem, the written presentation of the solution should be clear and the arguments easy to follow.

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1. Let X be a topological space, and let k be a field with the discrete topology. For every open U in X define
$$\mathcal{O}_X(U) = \{f : U \to k \mid f \text{ is continuous}\}.$$
   (a) Show that $\mathcal{O}_X$ is a sheaf of rings, with the obvious restriction maps, and compute the stalks $\mathcal{O}_{X,x}$.

   (b) Compute the tangent space of X at any point x when X is the circle S^1 with the metric topology and with the structure sheaf given above.

(7)

2. Let $S = k \times k \setminus \{(0,0)\}$, and say that $(x,y) \equiv (x',y')$ if $xy' = x'y$. Define the *projective line* $\mathbb{P}^1_k$ by $S/\equiv$. Let $\Phi, \Psi : k \to \mathbb{P}^1_k$ be defined by $\Phi(x) = (x,1)$ and $\Psi(x) = (1,x)$, for all $x \in k$ and let $U = \operatorname{im}\Phi$ and $V = \operatorname{im}\Psi$. Let $\mathcal{U}$ be the subsets W of $\mathbb{P}^1_k$ such $\Phi^{-1}(W)$ and $\Psi^{-1}(W)$ are open.

   (a) Show that $\mathcal{U}$ is a topology on $\mathbb{P}^1_k$ and that Φ, Ψ are homeomorphisms.

   (b) Show that $\{(U,k,\Phi),(V,k,\Psi)\}$ is an atlas on $\mathbb{P}^1_k$, which defines $\mathbb{P}^1_k$ as an analytic manifold.

   (c) Show that the ring $\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(\mathbb{P}^1_{\mathbb{R}})$ of analytic functions on $\mathbb{P}^1_{\mathbb{R}}$ is isomorphic to the ring of analytic functions $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x)$.

   (d) Show that the ring $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}})$ of analytic functions on $\mathbb{P}^1_{\mathbb{C}}$ consists entirely of constant functions. (*Hint:* Use Liouville's Theorem.)

(7)