KTH Teknikvetenskap

SF2723 Topics in Mathematics - Matrix groups
Homework Assignment 9
2008-11-05

The solutions should be handed in no later than November 12, 2008. The final grade will be based upon the total score on the homework and on the oral exam. The total maximal score on the homework assignments is 200 and in order to pass, at least 100 is required.

In order to get full score on each problem, the written presentation of the solution should be clear and the arguments easy to follow.

---

1. Show that $\mathbb{R}^3$ is a Lie algebra under the cross-product and that it is naturally isomorphic to the Lie algebra of $\mathrm{SO}_3(\mathbb{R})$. **(4)**

2. Let H be the subset of $\mathrm{Gl}_{n+2}(\mathbb{C})$ defined by matrices of the form
$$\begin{pmatrix} 1 & {}^t x & y \\ 0 & I_n & z \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{where } x, z \in \mathbb{C}^n \text{ and } y \in \mathbb{C}.$$
   (a) Show that H is a Lie group.
   (b) Determine the Lie algebra $\mathfrak{h}$ of H.

   (4)

3. In quantum mechanics, we have the *Pauli spin matrices*
$$1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad \sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right) \quad \text{and} \quad \sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$
   (a) Show that the set $\{1, \sigma_x, \sigma_y, \sigma_z\}$ spans the Lie algebra $\mathfrak{gl}_2(\mathbb{C})$.
   (b) Show that the set $\{\sigma_x, \sigma_y, \sigma_z\}$ spans a three dimensional sub Lie algebra of $\mathfrak{gl}_2(\mathbb{C})$ which is identical to $\mathfrak{sl}_2(\mathbb{C})$.
   (c) Show that the Lie algebra of (b) is isomorphic to the Lie algebras of the first problem.

   (2,2,2)