3.6 Z-transform

Diskret version av Laplace transf.

För en sekvens $\{a_n\}_{n=0}^{\infty}$ definierar vi funktionen

$$A(z) = \sum_{n=0}^{\infty} a_n z^{-n}$$ som Z-transf. av serien

(Detta kallas även Laurentserie)

[Anm. om $f(x) = \sum_{n=0}^{\infty} a_n \delta_n(x)$ dvs. $f(n) = a_n$ och $f = 0$ annars)

$$\Rightarrow \int_0^{\infty} f(x) e^{-xt} dx = \sum a_n \int e^{-xt} \delta_n(x) = \sum_{n=0}^{\infty} a_n e^{-nt}$$

$$= \sum_{n=0}^{\infty} a_n (e^t)^{-n} = \sum_0^{\infty} a_n z^{-n} \quad ; \; e^t =: z)$$

Ex. $a_n = 1 \;\; \forall n \geq 0 \Rightarrow$

$$A(z) = \sum_0^{\infty} z^{-n} \underset{\substack{\uparrow \\ \text{geom.} \\ \text{serie}}}{=} \frac{z}{z-1}$$

ex. Låt $a_0 = 1$; $a_1 = 2$ och

$$a_{n+2} = 3a_{n+1} - 2a_n \quad n = 0, 1, 2 \ldots$$

Bestäm a_n $\forall n$.

lös multipl. med z^{-n} och addera

$$\sum_{n=0}^{\infty} a_{n+2} z^{-n} = 3 \sum_{n=0}^{\infty} a_{n+1} z^{-n} - 2 \sum_{0}^{\infty} a_n z^{-n}$$

sätt $A(z) := \sum_{0}^{\infty} a_n z^{-n} = \underbrace{1 + \frac{2}{z}}_{givna} + \frac{a_2}{z^2} + \cdots$

$$\sum_{0}^{\infty} a_{n+1} z^{-n} = \{n+1 = k\} = \sum_{k=1}^{\infty} a_k z^{-(k-1)} =$$

$$= z \underbrace{\sum_{k=1}^{\infty} a_k z^{-k}}_{A(z) - \underset{1}{a_0}} = z\left(A(z) - 1\right)$$

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$$\sum_{n=0}^{\infty} a_{n+2} z^{-n} = z^2 \underbrace{\sum_{k=2}^{\infty} a_k z^{-k}}_{A(z) - a_0 - \frac{a_1}{z}} =$$

$$= z^2 \left(A(z) - 1 - \frac{2}{z} \right)$$

så ekv. blir

$$\cancel{\left(A(z) - 1\right)} = 3$$

$$z^2 \left(A(z) - 1 - \frac{2}{z} \right) = 3 \cdot z \left(A(z) - 1 \right) - 2A(z)$$

$$\Rightarrow \quad A(z) = \frac{z}{z-2} = \frac{1}{1 - (2/z)} = \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$= \sum_{0}^{\infty} 2^n z^{-n}$$

$$\boxed{a_n = 2^n}$$

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Ex. $a_n = \{n!\}_{n=1}^{\infty}$

$$A(z) = \sum_0^{\infty} (n!)\, z^{-n} \quad \text{divergerar } \forall z$$

$$|z| = \sigma \Rightarrow \quad \sum n!\, |z|^{-n} = \sum n!\, \sigma^{-n} = \infty$$

$$\frac{(n+1)!\, \sigma^{-n-1}}{n!\, \sigma^{-n}} = (n+1)\, \sigma^{-1} \to \infty$$

Sats 3.7

$$\mathcal{Z}(\lambda_1 a + \lambda_2 b) = \lambda_1 \mathcal{Z}(a) + \lambda_2 \mathcal{Z}(b)$$

$$\mathcal{Z}(\lambda^n a_n) = A\left(\frac{z}{\lambda}\right)$$

$$\mathcal{Z}(a_{n+k}) = z^k \left(A(z) - a_0 - \frac{a_1}{z} - \dots - \frac{a_{k-1}}{z^{k-1}}\right)$$

$$\mathcal{Z}(a_{n-k}) = z^{-k} A(z) \qquad k > 0$$

$$\mathcal{Z}(n a_n) = -z A'(z)$$

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Ex. Fibonaccitalen

$f_0 = f_1 = 1; \quad f_{n+2} = f_{n+1} + f_n \quad n \geq 0$

$F = Z(f)$; satsen ger

$$Z(f_{n+2})_{n=0}^{\infty} = z^2\left(F(z) - a_0 - \frac{a_1}{z}\right) = z^2\left(F(z) - 1 - \frac{1}{z}\right)$$

$$Z(f_{n+1})_{n=0}^{\infty} = z\left(F(z) - a_0\right) = z\left(F(z) - 1\right)$$

$$z^2\left(F(z) - 1 - \frac{1}{z}\right) = z\left(F(z) - 1\right) + F(z)$$

$$z^2F - z^2 - z = zF - z + F$$

$$F = \frac{z^2}{z^2 - z - 1} = \frac{Az}{z - \alpha} + \frac{Bz}{z - \beta}$$

där $\alpha = \frac{1+\sqrt{5}}{2}$; $\beta = \frac{1-\sqrt{5}}{2}$

$$A = \frac{\sqrt{5}+1}{2\sqrt{5}} \quad ; \quad B = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$F = A \sum \alpha^n z^{-n} + B \sum \beta^n z^{-n}$$

$$= \sum_0^{\infty} \underbrace{\left(A\alpha^n + B\beta^n\right)}_{f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]} z^{-n}$$

$n = 0, 1, 2, \ldots$

Convolution (faltning)

a, b är $\mathcal{Z}$-serier så är

c : $\quad c_n = \sum_{k=0}^{\infty} a_{n-k} b_k = \sum_{k=0}^{n} a_k b_{n-k}$

$n = 0, 1, 2, \ldots$

$c = a * b$

$C(z) = \mathcal{Z}(c) = ?$

$$\mathcal{Z}(c) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_{n-k} b_k \right) z^{-n} =$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{n-k} \cdot b_k z^{-n} =$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{n-k} z^{-(n-k)} b_k z^{-k}$$

$$= \underbrace{\sum_{k=0}^{\infty} b_k z^{-k}}_{B(z)} \Bigg(\underbrace{\sum_{n=k}^{\infty} a_{n-k} z^{-(n-k)}}_{\underbrace{\sum_{m=0}^{\infty} a_m z^{-m}}_{A(z)}} \Bigg) = B(z) A(z)$$

3.48 Bestäm a_n; $n=0,1,2..$ sådana att (7)

Övn $a_0=a_1=0$; $a_{n+2}-3a_{n+1}+2a_n=1-2n$

$n=0,1,2..$

Lös multip. med z^{-n} och addera

$$\sum_{n=0}^{\infty} a_{n+2}z^{-n} - 3\sum_0^{\infty} a_{n+1}z^{-n} + 2\sum_0^{\infty} a_n z^{-n} = \sum_0^{\infty}(1-2n)z^{-n}$$

$\Rightarrow$ låt $A(z)=\sum a_n z^{-n}$

då får vi

$$z^2\Big(A(z)-\underbrace{a_0-\frac{a_1}{z}}_{0}\Big) - 3z\big(A(z)-\underbrace{a_0}_{=0}\big) + 2A(z) = \sum(1-2n)z^{-n}$$

$$A(z)\left(z^2-3z+2\right) = \underbrace{\sum z^{-n}}_{\frac{z}{z-1}} - 2\underbrace{\sum_{n=0}^{\infty} nz^{-n}}_{\cancel{+}\,\cancel{2}z\left(\frac{z}{z-1}\right)'}$$

(V) sats 3.7

$$A(z)=\frac{\frac{z}{z-1}+2z\cdot\left(\frac{-1}{(z-1)^2}\right)}{z^2-3z+2} = \frac{\cancel{3z}\; z^2-3z}{(z-1)\underbrace{(z^2-3z+2)}_{(z-2)(z-1)}} =$$

$$A(z) = \frac{\cancel{z(z+1)}\; z^2-3z}{(z-1)^3(z-2)} = \frac{A}{(z-1)^3}+\frac{B}{(z-1)^2}+\frac{C}{(z-1)}+\frac{D}{(z-2)}$$

$$= \cancel{\frac{4}{(z-1)^2}+\frac{2}{z-1}+\frac{-2}{z-2}}$$

$$A(z) = \frac{2}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{2}{z-1} - \frac{2}{z-2} \quad (8)$$

$$\frac{\cancel{z}}{z-1} = \sum_{n=0}^{\infty} z^{-n} \quad , \quad \frac{1}{z-1} = \sum_{n=0}^{\infty} z^{-n-1} \underset{n+1=m}{\overset{\uparrow}{=}} \sum_{m=1}^{\infty} z^{-m}$$

$$\frac{1}{(z-1)^2} = -\left(\frac{1}{z-1}\right)' = + \sum_{n=1}^{\infty} + n z^{-n-1} \underset{n+1=m}{\overset{\uparrow}{=}} \sum_{m=2}^{\infty} (m-1) z^{-m}$$

$$\frac{2}{(z-1)^3} = \left(\frac{1}{z-1}\right)'' = -\sum_{n=2}^{\infty} (n-1)n\, z^{-n-1} = -\sum_{n=3}^{\infty} (n-2)(n-1) z^{-n}$$

$$\frac{1}{z-2} = \cancel{}\ \frac{1}{z} \cdot \frac{z}{z-2} = \frac{1}{z} \sum_{n=0}^{\infty} 2^n z^{-n} = \sum_{n=1}^{\infty} 2^{n-1} z^{-n}$$

$$\Rightarrow A(z) = \cancel{-\sum_{n=3}^{\infty} (n-2)(n-1) z^{-n}}$$

$$= \sum_{n=3}^{\infty} \left[-(n-2)(n-1) + 3(n-1) + 2 - 2 \cdot 2^{n-1}\right] z^{-n}$$

$$+ \underbrace{(3\ \cancel{2} + 2 - 2 \cdot 2)}_{z^{-2}} z^{-2} + \underbrace{(2-2)}_{0} z^{-1} + 0\cancel{z}$$

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$$A(z) = \sum_{n=3}^{\infty} \Big[\underbrace{(n-1)(5-n) + 2 - 2^n}_{1 \text{ då } n=2}\Big] z^{-n} + z^{-2}$$

$$= \sum_{n=2}^{\infty} \Big[(n-1)(5-n) + 2 - 2^n\Big] z^{-n}$$

3.7 Tillämpning i kontrollteori

<Själv s>

8.9 Fundamental lösning

$$P(r) = \sum_{j=0}^{n} a_j r^j \quad ; \quad \text{polyn.}$$

$$P(D) = \sum a_j D^j \quad \text{operator}$$

$$P(D)y = \sum a_j y^{(j)}$$

$$P(D)(f * \varphi) = (P(D)f) * \varphi = f * (P(D)\varphi)$$

om E : $P(D)E = \delta \implies$

$$P(D)(E * f) = \cancel{(P(D)E, f) = (\delta, f) = f}$$

$$= (P(D)E) * f = \delta * f = f$$

E är en fundam. lösning

Ex. $P(D) = D^2 + a^2$, bestäm en lösning till

$E'' + a^2 E = \delta$; F-transform ger

$$(i\omega)^2 \hat{E} + a^2 \hat{E} = \hat{\delta} = 1$$

$$\hat{E} = \frac{1}{-\omega^2 + a^2} = \frac{i}{4a}\left(\frac{2}{i(\omega+a)} - \frac{2}{i(\omega-a)}\right)$$

obs! $\mathcal{F}(\operatorname{sgn} t) = \frac{2}{i\omega}$

$\Downarrow$ $\mathcal{F}(e^{\pm ait}\operatorname{sgn} t) = \frac{2}{i(\omega \mp a)}$

$$y = \frac{i}{4a}\left(e^{-iat}\operatorname{sgn} t - e^{iat}\operatorname{sgn} t\right) = \frac{1}{2a}\sin at\,(2H(t)-1)$$