8.1–8.4 Distributioner

δ – Dirac = ?? ?? positiv ??

Test flkner: C^∞-flkner

$$\text{support}(\varphi) = \overline{\{\varphi(x) \neq 0\}}$$

= höljet av icke noll ~~??~~ pkt

Schwartz class: $\mathcal{S}$

$$\left|(1+|x|^n)\,\varphi^{(k)}(x)\right| \leq C_{n,k} \qquad \forall\, n,k \geq 0$$

Funktionen

$$\varphi(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases} = e^{-1/x} H(x)$$

$\varphi^{(k)}(x)$ exist. $\forall k$

& $\varphi^{(k)}(0) = 0 \quad \forall k$

Läs ex. 8.10

Distributioner (Tempererade)

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$f: S \longrightarrow \mathbb{C}$ = komplexa tal

1) f linjär

2) f kontin.

mängden temp. distr. betecknas S'

<u>ex.</u> 1) $\delta_a : S \longrightarrow \mathbb{C}$ Temp.

$\varphi \longrightarrow \varphi(a)$

2) $\delta_a' : S \longrightarrow \mathbb{C}$ Temp

$\varphi \longrightarrow -\varphi'(a)$

3) $f: S \longrightarrow \mathbb{C}$ Ej temp.

$\varphi \longrightarrow \int_0^\infty e^{x^2}\varphi(x)$

då $\varphi = e^{-x^2}$ ger $\int \cdots$ = div.

Distrib. ~~...~~ är egentligen generaliserade (3)

funktioner (ej punktvis defin.)

Läs ex. i boken; speciellt 8-15

övn. visa att $f = \ln|x|$ är en temp distr

$f : \mathcal{S} \to \mathbb{C}$ är linjär

$$\int f \varphi_j = \int \ln|x| \, \varphi_j \qquad ; \ \varphi_j \to \varphi$$

$$\int |\ln|x|| \, |\varphi_j - \varphi| \le \cancel{\int \underbrace{\frac{\ln|x|}{|x|}}_{\le 1} |\varphi_j - \varphi| \, |x|}$$

$$= \underbrace{\int_{-1}^{1} [\ldots]}_{\le \left(\int_{-1}^{1} |\ln|x||\right) \underbrace{\sup|\varphi_j - \varphi|}_{\to 0}} + \underbrace{\int_{|x|>1} \ln|x| \, (\varphi_j - \varphi)}_{\le \int_{|x|>1} \frac{\ln|x|}{|x|^2} |\varphi_j - \varphi|}$$

$\int_{-1}^{1} |\ln|x||$: ändlig

$$\le \int_{|x|>1} \frac{1}{|x|^2} \cdot \max \ldots$$

§4 Egenskaper

Derivata: $f \in S'$

$$\Rightarrow \quad f'(\varphi) := -f(\varphi') \qquad \forall \varphi \in S$$

f' = deriv. av f

ex. $H'(\varphi) = -H(\varphi') = \delta(\varphi)$

$$\delta'(\varphi) = -\delta(\varphi') = -\varphi'(0)$$

$$\delta^{(n)}(\varphi) = (-1)^n \varphi^{(n)}(0)$$

produktregel: $(fg)' = f'g + fg'$ om f multip. g distrib.

ex. $f(x) = |x^2-1| = (x^2-1)\, H(x-1) + (1-x^2)\left[H(x+1) - H(x-1)\right]$

$$+ (x^2-1)\,(1 - H(x+1))$$

$$= (x^2-1)\left[2H(x-1) - 2H(x+1) + 1\right]$$

$$f' = 2x[\ldots] + \underbrace{(x^2-1)\left[2\delta_{-1} - 2\delta_1\right]}_{=0} = 2x[\ldots]$$

Övn. visa $x^2\delta''' = 6\delta'$ ③

$$(x^2\delta''',\varphi) = \cancel{x\delta''}\,(\delta''', x^2\varphi) =$$

$$= -(x^2\varphi)'''(0) = -\cancel{[\varphi + x\varphi']''(0)}$$

$$\cancel{= -[2\varphi' + x\varphi'']' = -[2\varphi'' + \varphi'' + x\varphi''']}(0)$$

$$= -\left[2x\varphi + x^2\varphi'\right]''(0) = -\left[2\varphi + 4x\varphi' + x^2\varphi''\right]'(0)$$

$$= -\left[6\varphi' + 6x\varphi'' + x^2\varphi'''\right](0) = -6\varphi'(0)$$

$$= (6\delta', \varphi)$$

8.8 Visa att om $f = \text{P.V.}\,\dfrac{1}{x}$ så är (6

$$f'(\varphi) = -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi(x)-\varphi(0)}{x^2}\,dx$$

Lös.

$$f(\varphi) := \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x} = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}$$

$$f'(\varphi) = -f(\varphi') = -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi'(x)}{x} =$$

$$= -\lim_{\varepsilon\to 0}\left[\left[\frac{\varphi(x)}{x}\right]_{-\infty}^{-\varepsilon} + \left[\frac{\varphi(x)}{x}\right]_{\varepsilon}^{\infty} + \int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}\right.$$

$$\underbrace{+\frac{\varphi(-\varepsilon)}{-\varepsilon} - \frac{\varphi(\varepsilon)}{\varepsilon}}_{\to 0}$$

exist ?

$$\frac{-\varphi(-\varepsilon)-\varphi(\varepsilon)}{\varepsilon}$$

existerar inte

Ett nytt försök

$$-f(\varphi') = -\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi'(x)}{x} = -\lim_{\varepsilon\to 0}\int\frac{[\varphi(x)-\varphi(0)]'}{x}$$

$$= -\lim_{\varepsilon\to 0}\left[\left(\frac{\varphi(x)-\varphi(0)}{x}\right)_{-\infty}^{-\varepsilon} + \left(\cdots\right)_{\varepsilon}^{\infty} + \int_{|x|>\varepsilon}\frac{\varphi(x)-\varphi(0)}{x^2}\right]$$

$$= \underbrace{-\lim_{\varepsilon\to 0}\left[\frac{\varphi(-\varepsilon)-\varphi(0)}{-\varepsilon}\right]}_{\varphi'(0)} \underbrace{-}_{-} \underbrace{\lim_{\varepsilon\to 0}\left[\frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon}\right]}_{\varphi'(0)}$$

$$\underbrace{+\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi(x)-\varphi(0)}{x^2}}_{\text{Svar}} = \cancel{\left(\mathrm{p.v.}\frac{1}{x^2},\varphi\right)}$$

Övn. vad f'' då $f = pv(\frac{1}{x})$

$\cancel{\text{om } g = f(x) =}$

$$f''(\varphi) = -f'(\varphi') = -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi'(x)-\varphi'(0)}{x^2} \quad ?$$

$\varphi'(x) -$

$\cancel{\langle f, g\rangle = \int fg \; G -}$

8.5 Four. Transfor

$\widehat{\varphi^{(k)}}(\omega) = (i\omega)^k \hat{\varphi}(\omega)$; $\widehat{\varphi^{(k)}}$ begr. $\Rightarrow$

$|\hat{\varphi}| \le |\omega|^{-k} M_k$ $\forall k$

R.S.S. $\quad \dfrac{d^n}{d\omega^n}\hat{\varphi} = \mathcal{F}((-ix)^n \varphi) \Rightarrow$ C^∞ och vi får $|(\hat{\varphi})^{(n)}| \le |\omega|^{-k} M_{k,n}$

so om $\varphi \in S \Rightarrow \hat{\varphi} \in S$

~~och att $\hat{\varphi} \to 0$ i ∞~~

$S \overset{\wedge}{\longleftrightarrow} S$ bijektiv

Def. $\quad \hat{f}(\varphi) = f(\hat{\varphi})$ $\quad f$ distrib.

$\hat{f}$ är en tempererad distrib.

ex. $\quad f \equiv 1 \Rightarrow \hat{f} = ?$

$$\hat{f}(\varphi) = f(\hat{\varphi}) = \int_{R^n} 1 \cdot \hat{\varphi} = 2\pi \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi} e^{i \cdot 0 x}}_{\varphi(0) \text{ inversion form}} = 2\pi\varphi(0)$$

so $\hat{f} = 2\pi\delta_0$

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<u>ex.</u> $f = x^n$

$$\hat{f}(\varphi) = f(\hat{\varphi}) = \int_{-\infty}^{\infty} x^n \hat{\varphi}(x)\,dx =$$

$$= \int x^n \hat{\varphi}(x) e^{-i\cancel{t}x}\,dx \Big|_{\cancel{t}=0} = \left[\frac{d^n}{\cancel{d\omega^n}\; d(-i\cancel{t})^n} \int \hat{\varphi}(x) e^{-i\cancel{t}x}\,dx\right]_{\cancel{t}=0}$$

$$= \cancel{e^{i\omega}(-i)^n \frac{d^n}{d\omega^n}\hat{\varphi}\Big|_{\omega=0} = (-i)^n (\hat{\varphi})^{(n)}(0) = (-i\delta^{(n)}}$$

$$= \cancel{(-i)^n \delta^{(n)}}$$

$$= (-i)^n \left[\frac{d^n}{d\cancel{t}^n} \underbrace{\int \hat{\varphi}(x) e^{-i\cancel{t}x}}_{2\pi\varphi(\cancel{t})}\right]_{\cancel{t}=0}$$

$$= (-i)^n \cdot 2\pi \cdot \varphi^{(n)}(0) = 2\pi(+i)^n \delta_0^{(n)}$$

$$\underbrace{(-1)^n \varphi^{(n)}(0)}_{\delta^{(n)}}$$

Sats: 8.3, egenskaper.

ex. $f = P.V\, \frac{1}{x}$

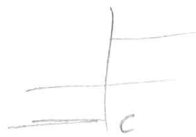
$\hat f = ?$; $\Rightarrow$ $xf = 1$ $\Rightarrow$ $\mathcal{F}(xf) = \mathcal{F}(1) = 2\pi\delta$

$\Rightarrow$ $i\frac{d}{d\omega}\hat f = 2\pi H'$

$D(\hat f + 2\pi i H) = 0$

sats 8.1 $\Rightarrow$ $\hat f + 2\pi i H = \text{konst}$

$\hat f = C - 2\pi i H$



~~$C - 2\pi i$~~

~~$C = C - 2\pi i$~~

f udda $\Rightarrow$ $\hat f$ udda $\overset{\text{visa}}{\Rightarrow}$ $C = \pi i$

$\boxed{\begin{aligned} &\hat f(-\omega) = -\hat f(\omega) \\ &C - 2\pi i H(-\omega) = -(C - 2\pi i H(\omega)) \\ &\Rightarrow \\ &2C = 2\pi i\,[\underbrace{H(-\omega) + H(\omega)}_{=1}] \\ &C = \pi i \end{aligned}}$

$\hat f = \pi i(1 - 2H)$

så $\hat f = -\pi i\, sgn(x)$

ex. 8.30 själva

~~$\mathcal{F}(P.V.\frac{1}{x}) = (\omega)$~~

$\hat H = \pi\delta - i\,P.V(\frac{1}{\omega})$

8.6 konvolution (convolution)

$$f * g = \int f(x-y) g(y)\, dy$$

$f \in S'$, $\varphi \in S$

$$f * \varphi (x) = \int f(y)\, \varphi(x-y)\, dy$$

eller det korrekta $= f * \varphi = f_y(\varphi(x-y))$

$= f(\varphi(x-\cdot))$ (rättare)

sätt $\quad \psi_h(x,y) = \dfrac{\varphi(x+h-y) - \varphi(x-y)}{h} \xrightarrow[h \to 0]{} \varphi'(x-y)$

$$\frac{f * \varphi(x+h) - f * \varphi(x)}{h} =$$

$$\frac{f(\varphi(x+h-\cdot)) - f(\varphi(x-\cdot))}{h} \underset{\text{lin.}}{=} f(\psi_h(x,\cdot))$$

$$\to f(\varphi'(x-\cdot)) = f * \varphi'(x)$$

$$D^n(f * \varphi) = f * \varphi^{(n)}$$

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$$\mathcal{F}(\varphi * f)(\psi) \underset{\substack{\uparrow \\ \text{def}}}{=} \varphi * f(\hat{\psi}) = \int \varphi * f \,\hat{\psi}\, dx$$

$$= \iint \varphi(x-y)\, f(y)\, \hat{\psi}(x)\, dx\, dy$$

$$\hat{\varphi}\hat{f}[\psi] \underset{\substack{\uparrow \\ \text{def} \\ \text{multiplik}}}{=} \hat{f}(\hat{\varphi}\psi) = f(\widehat{\hat{\varphi}\psi}) = f\left(\frac{1}{2\pi}\hat{\hat{\varphi}} * \hat{\psi}\right)$$

$$f(\check{\varphi} * \hat{\psi}) = f_x\left(\int \varphi(y-x)\hat{\psi}\right) = \iint f(x)\,\varphi(y-x)\,\hat{\psi}$$

$\uparrow$

$\check{\varphi}(x) = \varphi(y-x)$

så $\widehat{\varphi * f} = \hat{\varphi}\hat{f}$

Obs! $\delta * f(x) = f(x)$ för f kont

8.7. själva