Dirichlet problem
på skivan

Laplace ekvationen

$\Delta u = 0$ i Ω

$u = g$ on $\partial\Omega$

Bestäm en lösning!

För $\Omega = \{|x|<1\}$ borde vara

enkelt!

Komplexa planet: $z = re^{i\theta}$

$u = u(re^{i\theta})$ $r<1$

u harmonisk: $\Delta u = 0$

$u = g$ på $\partial D = \{|x|=1\}$

dvs

$$\lim_{r \to 1} u(re^{i\theta}) = g(x)$$

eller $\tilde{g}(\theta) = g(x)$ så är

$$\lim_{r \to 1} u(re^{i\theta}) = \tilde{g}(\theta)$$

ex. $g(x) = x + x^2 + y^2$

då är $\tilde{g}(\theta) = \cos\theta + 1 \quad -\pi < \theta \leq \pi$

ex. $g(x) = x^4 + y^4$

$\tilde{g}(\theta) = \cos^4\theta + \sin^4\theta \qquad -\pi < \theta \leq \pi$

Låt $f(z) = u(z) + iv(z)$

där v är harmonisk konjugat

skriv om
$$f(z) = \sum_{n=0}^{\infty} A_n r^n e^{in\theta}$$

$$\left\{ \begin{array}{l} A_n = B_n + iC_n \\ e^{i\theta} = \cos n\theta + i\sin n\theta \end{array} \right\}$$ ger

$$f(z) = \sum_{n=0}^{\infty} r^n (B_n \cos n\theta - C_n \sin n\theta) +$$
$$+ i \sum_{n=0}^{\infty} r^n (C_n \cos n\theta + B_n \sin n\theta)$$

så $$u = \sum_{n=0}^{\infty} r^n (B_n \cos n\theta - C_n \sin n\theta)$$

omskriv. ger

$$u = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

Metod 2 separation av variabler

polära koordinater ger

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial \theta^2}=0$$

om $u=R(r)\Theta(\theta)$

$\Rightarrow$

$$r\,\partial_r\left(r\,\partial_r u\,\Theta(\theta)\right)+R(r)\frac{d^2\Theta}{d\theta^2}=0$$

$\Rightarrow$ $r(rR')'\Theta=-R\Theta''$

$$\frac{1}{R}r(rR')'=-\frac{\Theta''}{\Theta}=\lambda$$

$$\Theta''+\lambda\Theta=0$$

$$r(rR')'-\lambda R=0$$

Randdata:

$$\Theta'' + \lambda\Theta = 0$$

$$\Theta(2\pi + \theta) = \Theta(\theta) \quad \forall\, \theta$$
(periodisk)

$\lambda < 0$, $\lambda = 0$, $\lambda > 0$ ger

$\Theta_0 = A_0$; $\lambda < 0$ ingen periodisk lösn.

$\lambda = \omega^2$: $\Theta(\theta) = A_\omega \cdot \cos(\omega\theta) + B_\omega \sin(\omega\theta)$

2π-periom om $\omega = n$; $\lambda = n^2$

$\lambda = 0$ $\quad r(rR')' = 0 \implies R = C\ln r + D$

men $R(0^+)$ exist. om $C = 0$

$\lambda > 0$: $\lambda = n^2 \implies r(rR')' = n^2 R \implies$

$$r^2R'' + rR' - n^2R = 0 \qquad \underline{r = e^s} \implies$$

$$\frac{d^2R}{ds^2} - n^2R = 0 \Rightarrow R = Cr^n + Dr^{-n}$$

$R(0^+) =$ exist $\Rightarrow \quad D = 0$

$$U = \theta_n R_n = r^n(a_n \cos n\theta + b_n \sin n\theta)$$

och vi får formalt

$$U = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

Poisson kärnan: Skriv om lösningen

i komplex form

$$U = \sum_{n \in \mathbb{Z}} c_n r^{|n|} e^{in\theta}$$

då $r \to 1 \Rightarrow \quad u(re^{i\theta}) \to g(\theta)$

$\Rightarrow$

(7)

dvs $\quad c_n = \text{Fourier koeff}(g(\theta))$

$$= \frac{1}{2\pi}\int_{T} g(\theta)\, e^{-in\theta}\, d\theta$$

$$\Rightarrow \quad u(r,\theta) = \sum_{\mathbb{Z}} r^{|n|} e^{in\theta} \left(\frac{1}{2\pi} \int_T g(t) e^{-int} \right)$$

$$= \int_T \underbrace{\underbrace{\left(\frac{1}{2\pi} \sum_{\mathbb{Z}} r^{|n|} e^{in(\theta-t)} \right)}_{\frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}}}_{P_r(\theta-t) \text{ poisson kärna}} g(t)$$

$$= \int_T P_r(\theta-t)\, g(t)\, dt$$

Visa att $P_r(s)$ är

en poisson kärna?

$$\begin{cases} P_r(s) = P_r(-s) \geq 0 & \forall s \in T \quad r < 1 \\ \int_T P_r(s)\,ds = 1 & r < 1 \\ \lim_{r \to 1} \int_\delta^\pi P_r(s)\,ds = 0 & \end{cases}$$

ex. 6.6 Bestäm u: $\Delta u = 0$ i D

$u = \cos 4\theta - 1$ på ∂D

Lös. $u = -1 + r^4 \cos 4\theta$

skriv om i x, y variabler

$u = 1 + x^4 - 6x^2y^2 + y^4$

(utveckla $\cos 4\theta = \ldots \cos\theta, \sin\theta$, produkt av dem)

Övn. 6.12, Bestäm u harmonisk i Δ; samt $u = 2 + \cos 3\theta + \sin 4\theta$ på $\partial\Delta$

Lös. Låt $g_1 = 2$; $g_2 = \cos 3\theta$, $g_3 = \sin 4\theta$

u_i ; g_i $i = 1, 2, 3$

$u_1 = 2$ en lösn till $\underline{\Delta u_1 = 0; \ u_1 = 2 \ \partial\Delta}$

$u_2 = r^3 \cos 3\theta = \mathrm{Re}\left((re^{i\theta})^3\right) = \mathrm{Re}(z^3)$

z^3 är analytisk så $\mathrm{Re}(z^3)$ harm

då $r = 1 \Rightarrow u_2 = \cos 3\theta$

p.s.s.
$u_3 = r^4 \sin 4\theta$ en lösn för g_3

$$\boxed{u = 2 + r^3 \cos 3\theta + r^4 \sin 4\theta}$$

you can rewrite it in terms of x, y

6.13 $\Delta u = 0,$ in D $\quad u = x^4 + y^4$ on ∂D

sol: rewrite $g = x^4 + y^4 \Rightarrow$

$$x^4 + y^4 = (x^2+y^2)^2 - 2x^2y^2 = \text{etc...}$$

$$g(\theta) = \cos^4\theta + \sin^4\theta = \text{use trig formula}$$

$$= (\cos^2\theta + \sin^2\theta)^2 - 2\cos^2\theta\sin^2\theta$$

$$= 1 - 2(\cos\theta\sin\theta)^2 = 1 - 2\left(\frac{\sin 2\theta}{2}\right)^2$$

$$= 1 - \frac{1}{2}\sin^2 2\theta = 1 - \frac{1}{2}\left(\frac{1-\cos 4\theta}{2}\right)$$

$$= 1 - \frac{1}{4} + \frac{1}{4}\cos 4\theta = \frac{3}{8} + \frac{1}{8}\cos 4\theta$$

$$\Rightarrow u = \frac{3}{4} + \frac{1}{4} r^4 \cos 4\theta \quad \text{är en lösning}$$

$$= \frac{3}{4} + \frac{1}{4}\operatorname{Re}(z^4) = \frac{3}{4} + \frac{1}{4}\operatorname{Re}\left((x+iy)^4\right) =$$

$$= 3/4 + 1/4 \cdot \left[(x^4 + y^4 - 6x^2y^2\right]$$

6.14 $g = \sin^3\theta$ på ∂D

$$\sin^3\theta = \sin\theta \cdot \left[\frac{1-\cos 2\theta}{2}\right] =$$

$$= \frac{\sin\theta}{2} - \frac{\sin\theta\cos 2\theta}{2} =$$

$$= \frac{\sin\theta}{2} - \sin 3\theta - \sin\theta = -\frac{\sin\theta}{2} - \sin 3\theta$$

$$\Rightarrow \quad u = -\frac{r\sin\theta}{2} - r^3\sin 3\theta$$

$$= -\frac{1}{2}\,\mathrm{Im}(z) - \mathrm{Im}(z^3)$$

$$= -\frac{1}{2}\,y - \mathrm{Im}\left((x+iy)^3\right) =$$

$$= -\frac{1}{2}\,y - \cdots$$