SF2735 Homologisk Algebra
Exercise set 4

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The solutions to these exercises are to be handed in no later than Thursday, 26th of November. Please pay attention to the presentation as well as the arguments given in the solutions.

Exercise 1

(a). Let $\mathbf{C}$ be the filed of complex numbers and $\mathbf{C}^n := \{(z_1, \ldots, z_n) \mid z_i \in \mathbf{C}\}$. Recall that for $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, $|z| = \sqrt{z_1\overline{z_1} + z_2\overline{z_2} + \cdots z_n\overline{z_n}}$, where for a complex number $w = a+bi$, $\overline{w} = a - bi$. We say that a subset $U \subset \mathbf{C}^n$ is open if, for any $z \in U$, there is $r > 0$ such that $\{w \in \mathbf{C}^n \mid |w - z| < r\} \subset U$. Show that this defines a topology on $\mathbf{C}^n$. Prove that $\mathbf{C}^n$ with this topology is isomorphic to $\mathbf{R}^{2n}$ (the Euclidean space of dimension 2n).

(b). Define $S := \{z \in \mathbf{C}^n \mid |z| = 1\} \subset \mathbf{C}^n$. Show that S, with the subspace topology of $\mathbf{C}^n$, is isomorphic to S^{2n-1}.

(c). Let $n \geq 1$. Define $\mathbf{C}P^{n-1}$ to be the set of 1-dimensional $\mathbf{C}$ vector subspaces of $\mathbf{C}^n$. Let $\pi : S \to \mathbf{C}P^{n-1}$ be the function that assigns to an element $z \in S$, the $\mathbf{C}$ vector subspace spanned by z, i.e., $\pi(z) = \{cz \mid c \in \mathbf{C}\}$. The set $\mathbf{C}P^{n-1}$ together with the quotient topology induced by the function π is called the $(n-1)$-dimensional complex projective space. Show that it is a compact space.

(d). Prove that $\mathbf{C}P^n$ is isomorphic to $\mathbf{C}P^{n-1} \cup_\pi D^{2n}$. Conclude that $\mathbf{C}P^1$ is isomorphic to S^2.

Exercise 2

(a). Show that S^0 is not a path connected space. Describe $\pi_0(S^0)$.

(b). Let $\alpha : S^n \to X$ be a map. Show that if $n > 0$, then the sets $\pi_0(X)$ and $\pi_0(X \cup_\alpha D^{n+1})$ are isomorphic. Give an example when $\pi_0(X)$ and $\pi_0(X \cup_\alpha D^1)$ are not isomorphic.

(c). Prove that, for any $n \geq 0$, the spaces $\mathbf{C}P^n$ and $\mathbf{R}P^n$ are path connected.