SF2735 Homologisk Algebra
Exercise set 5

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The solutions to these exercises are to be handed in no later than Thursday, 17th of December. Please pay attention to the presentation as well as the arguments given in the solutions.

Exercise 1

(a). Let $\alpha : \Delta^1 \to X$ be a 1-dimensional singular simplex in X. Define $\beta : \Delta^1 \to X$ by the formula $\beta(t_0, t_1) := \sigma(t_1, t_0)$. Show that $\alpha + \beta$ is a boundary in $S_1(X)$.

(b). Let $\sigma : \Delta^1 \to S^1$ be the map given by $\sigma(t_0, t_1) := (\sin(2t_0\pi), \cos(2t_0\pi))$. Show that this singular simplex is a cycle. Show moreover that its homology class is a generator of $H_1(S^1) = \mathbf{Z}$.

(c). Let $\tau : \Delta^1 \to S^1$ be the map given by $\tau(t_0, t_1) := (\sin(2t_1\pi), \cos(2t_1\pi))$. Show that this singular simplex is a cycle. Show moreover that $\sigma + \tau$ is a boundary, where σ is the singular simplex defined in part b. Conclude that in $H_1(S^1)$, we have $[\tau] = -[\sigma]$.

(d). Use parts b and c to show that the map $f : S^1 \to S^1$, given by $f(x_1, x_2) := (x_1, -x_2)$, induces multiplication by -1 on $H_1(S^1)$.

(e). Let $\tau : \Delta^1 \to S^1$ be the map given by $\tau(t_0, t_1) := (\sin(4t_0\pi), \cos(4t_0\pi))$. Show that this singular simplex is a cycle and that its homology class $[\tau] \in H_1(S^1)$ is equal to $2[\sigma]$, where σ is the singular simplex defined in part b.

Exercise 2

(a). Let $n \geq 1$ and $f : D^n \to D^n$ be an isomorphism . Show that f has to map the sphere $S^{n-1} \subset D^n$ into the sphere $S^{n-1} \subset D^n$.

(b). Let $n \geq 1$. Let us choose an isomorphism $h : \Delta^n \to D^n$. Recall that S^n is isomorphic to $D^0 \cup_\alpha D^n$ (see Example 7.9.4 in the notes). Let $g : D^n \to S^n$ be the map given by the composition of the inclusion $D^n \subset D^0 \coprod D^n$ and the quotient map $D^0 \coprod D^n \to D^0 \cup_\alpha D^n = S^n$. Show that the composition $gh : \Delta^n \to S^n$ is a cycle if and only if n is odd. Prove that if n is odd, then the homology class $[gh]$ is a generator of $H_n(S^n) = \mathbf{Z}$.