FÖRSLAG TILL LÖSNING TENTAMEN I MATEMATIK II FÖR CL, SF1613. 2009-08-25

1.

$$\det A = \begin{vmatrix} 1 & x & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ x & 1 & 1 & 1 & 1 \\ x & 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & -1 & 0 \\ x & 1 & 0 & 1 & 0 \\ x & 0 & 0 & 0 & 1 \end{vmatrix} = \{\text{UTV. LÄNGS 5:E KOL}\} =$$

$$= (-1)^{5+5} \cdot 1 \begin{vmatrix} 1 & x & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & -1 \\ x & 1 & 0 & 1 \end{vmatrix} = \{\text{UTV. LÄNGS 3:E KOL}\} = (-1)^{3+3} \cdot 2 \begin{vmatrix} 1 & x & 1 \\ 1 & 2 & 1 \\ x & 1 & 1 \end{vmatrix} =$$

$$= 2(2 + x^2 + 1 - 2x - 1 - x) = 2(x^2 - 3x + 2) = 2(x-1)(x-2)$$

$$\det A = 0 \iff 2(x-1)(x-2) = 0 \Rightarrow x = 1,\ x = 2.$$

SVAR! $x = 1$, $x = 2$

2.

$$\begin{cases} ax + y = b \\ x + ay = a \end{cases} \iff \underbrace{\begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \iff A\bar{v} = \bar{u}$$

$$\det A = a^2 - 1 \qquad \det A = 0 \Rightarrow a = \pm 1$$

1) $a \neq \pm 1$ GER UNIK LÖSNING. MED CRAMERS REGEL (EX.VIS):

$$x = \frac{\begin{vmatrix} b & 1 \\ a & a \end{vmatrix}}{a^2 - 1} = \frac{ab - a}{a^2 - 1} = \frac{a(b-1)}{(a+1)(a-1)} \qquad b \text{ godtycklig}$$

$$y = \frac{\begin{vmatrix} a & b \\ 1 & a \end{vmatrix}}{a^2 - 1} = \frac{a^2 - b}{a^2 - 1} \qquad b \text{ godtycklig}$$

2) $a = 1$ $\begin{cases} x + y = b \\ x + y = 1 \end{cases}$ OM $b = 1 \Rightarrow x + y = 1$, $x = 1 - y$ $\begin{cases} x = 1 - t \\ y = t \end{cases}$

OM $b \neq 1$ SAKNAS LÖSN.

3) $a = -1$ $\begin{cases} -x + y = b \\ x - y = -1 \end{cases}$ $\begin{cases} x - y = -b \\ x - y = -1 \end{cases}$ OM $b = 1 \Rightarrow x = y - 1$ $\begin{cases} x = t - 1 \\ y = t \end{cases}$

$b \neq -1$ SAKNAS LÖSN.

SVAR! 1) $a \neq \pm 1$, b godt $\begin{cases} x = \dfrac{a(b-1)}{(a+1)(a-1)} \\ y = \dfrac{a^2 - b}{(a+1)(a-1)} \end{cases}$ 2) $a = 1$, $b = 1$ $\begin{cases} x = 1 - t \\ y = t \end{cases}$ 3) $a = -1$, $b = 1$ $\begin{cases} x = t - 1 \\ y = t \end{cases}$

$a = 1$, $b \neq 1$ LÖSN. SAKNAS; $a = -1$, $b \neq 1$ LÖSN. SAKNAS

3. $\sum_{n=1}^{\infty} \frac{\sqrt[n]{e}-1}{\sqrt{n}}$ $a_n = \frac{\sqrt[n]{e}-1}{\sqrt{n}} = \frac{e^{\frac{1}{n}}-1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{e^{\frac{1}{n}}-1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{2n^2}+O(\frac{1}{n^3})-1}{\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{2n^2}+O(\frac{1}{n^3})}{\sqrt{n}} = \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{2n^2}+O(\frac{1}{n^3})\right)\cdot\frac{1}{\sqrt{n}}$$

VÄLJ $b_n = \frac{1}{n\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{1}{n\sqrt{n}}+\frac{1}{2n^2\sqrt{n}}+O(\frac{1}{n^3\sqrt{n}})}{\frac{1}{n\sqrt{n}}} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n}}\left(1+\frac{1}{2n}+O(\frac{1}{n^2})\right)}{\frac{1}{n\sqrt{n}}}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{2n}+O(\frac{1}{n^2})\right) = 1 \qquad \sum_{n=1}^{\infty} b_n = \frac{1}{n^{3/2}} \text{ ÄR KONV}$$

TY $\sum_{n=1}^{\infty} \frac{1}{n^a}$ KONV DÅ $a>1$. (JMF $\int_1^{\infty} \frac{1}{x^a}\,dx$ KONV DÅ $a>1$)

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt[n]{e}-1}{\sqrt{n}}$ ÄR KONV. ENL. JÄMFÖRELSEKRIT.

<u>SVAR</u>: KONVERGENT

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4. $x f'_x + y f'_y = 0$ $\begin{cases} u = \frac{y}{x} \\ v = y \end{cases}$

$$f'_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}\cdot -\frac{y}{x^2} = -\frac{y}{x^2}\frac{\partial f}{\partial u}$$

$$f'_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{1}{x} + \frac{\partial f}{\partial v}\cdot 1 = \frac{1}{x}\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

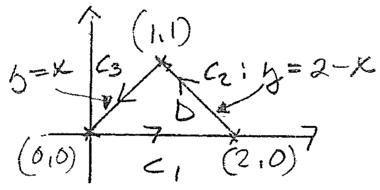
$x f'_x + y f'_y$ ÖVERGÅR NU TILL:

$$x\left(-\frac{y}{x^2}\frac{\partial f}{\partial u}\right) + y\left(\frac{1}{x}\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) = -\frac{y}{x}\frac{\partial f}{\partial u} + \frac{y}{x}\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} =$$

$$= y\frac{\partial f}{\partial v} = v\frac{\partial f}{\partial v} = 0, \quad \text{LÖSN: } \frac{\partial f}{\partial v} = 0 \Rightarrow f = g(u) = g\left(\frac{y}{x}\right)$$

<u>SVAR</u>: $f(x,y) = g\left(\frac{y}{x}\right)$ DÅR $g(t)$ ÄR EN ENVARIABELFUNKTION

5. $\int_C (y^2 - x^2)\,dx - 2xy\,dy$



SLUTET OMRÅDE, INGA SINGULÄRA PUNKTER
⇒ GREENS FORMEL KAN ANVÄNDAS.

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D -2y - (2y)\,dx\,dy =$$

$$= -\iint 4y\,dx\,dy = -4\int_0^1 \left(\int_y^{2-y} y\,dx\right) dy = -4\int_0^1 \left[yx\right]_y^{2-y} dy =$$

$$= -4\int_0^1 \left(y(2-y) - (y\cdot y)\right) dy = -4\int_0^1 (-2y^2 + 2y)\,dy =$$

$$= -4\left[-\frac{2y^3}{3} + y^2\right]_0^1 = -4\left(-\frac{2}{3} + 1\right) = -\frac{4}{3}.$$

SVAR: $-\frac{4}{3}$

6. $$\sum_{n=1}^{\infty} \frac{e^{n\ln n}\, x^n}{(2n+1)^n} = \sum_{n=1}^{\infty} \frac{e^{n\ln n}}{(2n+1)^n} x^n = \sum_{n=1}^{\infty} \frac{e^{\ln n^n}}{(2n+1)^n} x^n$$

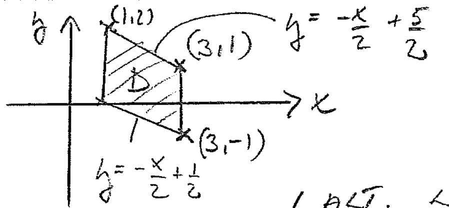
$$= \sum_{n=1}^{\infty} \frac{n^n}{(2n+1)^n} x^n = \sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n x^n = \sum_{n=1}^{\infty} a_n x^n$$

$a_n = \left(\frac{n}{2n+1}\right)^n$ KONV. RADIE $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} =$

$$= \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left(\frac{n}{2n+1}\right)^n}} = \lim_{n\to\infty} \frac{1}{\frac{n}{2n+1}} = \lim_{n\to\infty} \frac{2n+1}{n} = 2$$

SVAR: 2

7. $$\iint_D \frac{dx\,dy}{(1+x+2y)^2} =$$



$$= \int_1^3 \left(\int_{-\frac{x}{2}+\frac{1}{2}}^{-\frac{x}{2}+\frac{5}{2}} \frac{1}{(1+x+2y)^2}\,dy\right) dx =$$

(ALT. LÖSNING SÄTT $\begin{cases} u = x+2y \\ v = x \end{cases}$)

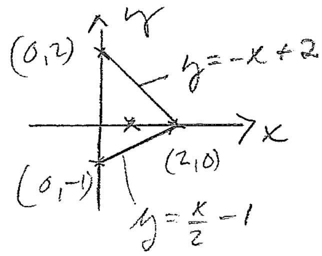
$$= \int_1^3 \left[\frac{-1}{1+x+2y}\cdot\frac{1}{2}\right]_{-\frac{x}{2}+\frac{1}{2}}^{-\frac{x}{2}+\frac{5}{2}} dx = \frac{1}{2}\int_1^3 \left(-\frac{1}{6} + \frac{1}{2}\right) dx = \frac{1}{2}\int_1^3 \frac{1}{3}\,dx = \frac{1}{6}\int_1^3 dx$$

$$= \frac{1}{6}\cdot 2 = \frac{1}{3}.$$

SVAR: $\frac{1}{3}$

8.

$$u(x,y) = 1 - x^2 + 2x - y^2$$



INRE STAT. PUNKTER:

$u'_x = -2x + 2 \qquad u'_x = 0 \Rightarrow x = 1$

$u'_y = -2y \qquad u'_y = 0 \Rightarrow y = 0$

(1,0) ÄR EN INRE STAT PUNKT. $u(1,0) = 2$

UNDERSÖK RANDEN:

$\underline{r_1}$: $x = 0$, $y = y$ $\qquad u(0,y) = 1 - y^2 = f(y)$

$f'(y) = -2y \qquad f' = 0 \Rightarrow y = 0 \qquad f(0) = 0$

$u(0,0) = 0$

$\underline{r_2}$: $y = \frac{x}{2} - 1 \qquad u\left(x, \frac{x}{2} - 1\right) = 1 - x^2 + 2x - \left(\frac{x}{2} - 1\right)^2 =$

$= 1 - x^2 + 2x - \frac{x^2}{4} + x - 1 = -\frac{5x^2}{4} + 3x = h(x)$

$h'(x) = -\frac{5}{2}x + 3 \qquad h'(x) = 0 \Rightarrow x = \frac{6}{5}$

$u\left(\frac{6}{5}, -\frac{2}{5}\right) = \frac{9}{5} \qquad h\left(\frac{6}{5}\right) = -\frac{5}{4} \cdot \frac{36}{25} + 3 \cdot \frac{6}{5} = -\frac{9}{5} + \frac{18}{5} = \frac{9}{5}$

r_3: $y = -x + 2 \qquad u(x, -x+2) = 1 - x^2 + 2x - (-x+2)^2 =$

$= 1 - x^2 + 2x - x^2 + 4x - 4 = -2x^2 + 6x - 3 = g(x)$

$g'(x) = -4x + 6 \qquad g'(x) = 0 \Rightarrow x = \frac{3}{2}$

$u\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{3}{2} \qquad g\left(\frac{3}{2}\right) = -2 \cdot \frac{9}{4} + 6 \cdot \frac{3}{2} - 3 = -\frac{9}{2} + 9 - 3 = 6 - \frac{9}{2} = \frac{3}{2}$

HÖRN: (0,−1) GER $u(0,-1) = -1$
(2,0) GER $u(2,0) = 1 - 4 + 4 = 1$
(0,2) GER $u(0,2) = 1 - 4 = -3$

<u>SVAR:</u> STÖRST VÄRDE: 2
MINST VÄRDE: −3

9.

$$A = \begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}$$

SÖK EGENVÄRDEN TILL A. KAR. EKV:

$$\begin{vmatrix} 2-\lambda & 2 & -5 \\ 3 & 7-\lambda & -15 \\ 1 & 2 & -4-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)(7-\lambda)(-4-\lambda) - 30 - 30 + 5(7-\lambda) + 30(2-\lambda) - 6(-4-\lambda) = 0$$

$$(2-\lambda)(7-\lambda)(-4-\lambda) - 60 + 35 - 5\lambda + 60 - 30\lambda + 24 + 6\lambda = 0$$

$$(2-\lambda)(-28 - 3\lambda + \lambda^2) - 29\lambda + 59 = 0$$

$$-56 - 6\lambda + 2\lambda^2 + 28\lambda + 3\lambda^2 - \lambda^3 - 29\lambda + 59 = 0$$

$$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \qquad \lambda = 1 \text{ ÄR EN ROT}$$

$$(\lambda-1)(\lambda^2 - 4\lambda + 3) = 0 \iff (\lambda-1)(\lambda-1)(\lambda-3) = 0$$

$\lambda = 1$ ÄR DUBBELROT.

SÖK EGENVEKTORER.

$$\lambda = 1 \quad \left(\begin{array}{ccc|c} 1 & 2 & -5 & 0 \\ 3 & 6 & -15 & 0 \\ 1 & 2 & -5 & 0 \end{array}\right) \cdot \tfrac{1}{3} \sim \left(\begin{array}{ccc|c} 1 & 2 & -5 & 0 \end{array}\right) \qquad \begin{array}{l} x + 2y - 5z = 0 \\ x = -2y + 5z \end{array}$$

$$\begin{array}{l} x = -2t + 5s \\ y = t \\ z = s \end{array} \qquad \bar{v}_1 = t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \qquad s, t \text{ GODT. HELA TAL}$$

$$\lambda = 3 \quad \left(\begin{array}{ccc|c} -1 & 2 & -5 & 0 \\ 3 & 4 & -15 & 0 \\ 1 & 2 & -7 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 2 & -5 & 0 \\ 0 & 10 & -30 & 0 \\ 0 & 4 & -12 & 0 \end{array}\right) \begin{array}{l} \\ \cdot \frac{1}{10} \\ \cdot \frac{1}{4} \end{array} \sim \left(\begin{array}{ccc|c} -1 & 2 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -1 & 2 & -5 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \qquad \begin{array}{l} x = 2y - 5z = 6t - 5t = t \\ y = 3z = 3t \\ z = t \end{array} \qquad \bar{v}_2 = t\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

VÄLJ $P = \begin{pmatrix} -2 & 5 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ $\det P \neq 0 \Rightarrow$ DET GÅR ATT

BILDA $P^{-1}AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

<u>SVAR!</u> A ÄR DIAGONALISERBAR. $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

10.

$$z = e^{xy}, \quad y = e^{xz} \qquad P = (0,1,1)$$

$$F_1: \ z - e^{xy} = 0 \qquad F_2: \ y - e^{xz} = 0$$

$$\bar{n}_1 = \text{grad}\, F_1 = (-ye^{xy}, -xe^{xy}, 1) \qquad \text{grad}\, F_1(0,1,1) = (-1,0,1)$$

$$\bar{n}_2 = \text{grad}\, F_2 = (-ze^{xz}, 1, -xe^{xz}) \qquad \text{grad}\, F_2(0,1,1) = (-1,1,0)$$

TANGENTVEKTORN $\bar{v}$ ÄR VINKELRÄT MOT BÅDE grad F_1 och grad F_2.

$$\bar{v} = \text{grad}\, F_1 \times \text{grad}\, F_2 = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = (0-1, -1-0, -1-0) = -(1,1,1) = k(1,1,1)$$

SVAR: $\bar{v} = k(1,1,1), \ k \in \mathbb{R}$

11.

$$I = \iiiint_V dx\,dy\,dz\,du \qquad x^2+y^2+z^2+u^2 \le 1$$

$$u^2 \le 1 - x^2 - y^2 - z^2$$

$$-\sqrt{1-x^2-y^2-z^2} \le u \le \sqrt{1-x^2-y^2-z^2}$$

$$I = \iiint \int_{-\sqrt{1-x^2-y^2-z^2}}^{\sqrt{1-x^2-y^2-z^2}} du\, dV = \iiint_V 2\sqrt{1-x^2-y^2-z^2}\, dx\,dy\,dz$$

$$= \left| \begin{array}{ll|l} \text{SFÄRISKA KOORDINATER} & & 1-x^2-y^2-z^2 = \\ x = r\sin\theta\cos\varphi & 0 \le r \le 1 & 1 - r^2\sin^2\theta\cos^2\varphi - r^2\sin^2\theta\sin^2\varphi - \\ y = r\sin\theta\sin\varphi & 0 \le \theta \le \pi & - r^2\cos^2\theta = 1 - r^2\sin^2\theta - r^2\cos^2\theta = \\ z = r\cos\theta & 0 \le \varphi \le 2\pi & = 1 - r^2 \\ dx\,dy\,dz = r^2\sin\theta\, dr\,d\theta\,d\varphi & & \end{array} \right|$$

$$I = 2\iiint \sqrt{1-r^2}\cdot r^2 \sin\theta\, dr\,d\theta\,d\varphi = 2\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \underbrace{\int_0^1 r^2\sqrt{1-r^2}\, dr}_{I_r}$$

$$I_2 = \int_0^1 r^2\sqrt{1-r^2}\, dr = \left| \begin{array}{ll} r = \sin t & 0 \le t \le \frac{\pi}{2} \\ dr = \cos t\, dt & \end{array} \right| = \int_0^{\frac{\pi}{2}} \sin^2 t \cdot \underbrace{\sqrt{1-\sin^2 t}}_{\cos^2 t} \cdot \cos t\, dt =$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 t\, |\cos t| \cos t\, dt = \int_0^{\pi/2} \frac{\sin^2 2t}{4}\, dt = \frac{1}{8}\int_0^{\pi/2} 1 - \cos 4t\, dt = \frac{1}{8}\left[t - \frac{\sin 4t}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{16}$$

$$I = 2 \cdot 2\pi \left[-\cos\theta\right]_0^{\pi} \cdot \frac{\pi}{16} = 4\pi(2)\cdot\frac{\pi}{16} = \frac{\pi^2}{2}.$$

SVAR: $\frac{\pi^2}{2}$