1.
$$\begin{cases} x+2y+z=0 \\ 3x-y-z=0 \\ 2x-3y+cz=0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & -3 & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$A\bar{x} = \bar{0}$$

HOMOGENT SYSTEM.

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & -3 & c \end{vmatrix} = -c-4-9+2-3-6c = -7c-14$$

$\det A = 0 \Leftrightarrow c=-2$. $c=-2$ GER OÄNDLIGT MÅNGA LÖSNINGAR.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & -3 & -2 & 0 \end{array}\right) \begin{matrix} (-3) & (-2) \\ & \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -7 & -4 & 0 \\ 0 & -7 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 7 & 4 & 0 \end{array}\right)$$

$$7y+4z=0 \qquad x+2y+z=0$$
$$y=-\frac{4z}{7} \qquad x=-2y-z=+\frac{8z}{7}-z=\frac{z}{7}$$

$$\begin{cases} x=\frac{t_1}{7} \\ y=-\frac{4}{7}t_1 \\ z=t_1 \end{cases} \Leftrightarrow \begin{cases} x=t \\ y=-4t \\ z=7t \end{cases}$$

SVAR: $c=-2$
$(x,y,z)=t(1,-4,7)$, $t\in\mathbb{R}$

2.
$$A=\begin{pmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad D=P^{-1}AP \Leftrightarrow A=PDP^{-1}$$
$$A^{1000}=PD^{1000}P^{-1}$$

SÖK A:S EGENVÄRDEN!
KAR. EKV!
$$\begin{vmatrix} 1-\lambda & -2 & 8 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)(-1-\lambda)^2=0$$
$$\lambda_1=1 \quad \lambda_{2,3}=-1$$

$$D=\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad D^{1000}=\begin{pmatrix} 1^{1000} & 0 & 0 \\ 0 & (-1)^{1000} & 0 \\ 0 & 0 & (-1)^{1000} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=E \qquad A^{1000}=PD^{1000}\cdot P^{-1}=P\cdot EP^{-1}=P\cdot P^{-1}=E$$

!! $A^{1000}=E=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

SVAR: $A^{1000}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(ALT2. $A^2=E$.
ALT3. $P=\begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ OCH $P^{-1}=\begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$)

3. $\sum_{k=1}^{\infty} \frac{1}{2k-\sqrt{k}}$ $\quad a_k = \frac{1}{2k-\sqrt{k}} = \frac{1}{k(2-\frac{1}{\sqrt{k}})}$ $\quad b_k = \frac{1}{k}$

$$\lim_{k\to\infty} \frac{\frac{1}{2k-\sqrt{k}}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{2k-\sqrt{k}} =$$

$$= \lim_{k\to\infty} \frac{k}{k(2-\frac{1}{\sqrt{k}})} = \lim_{k\to\infty} \frac{1}{2-\frac{1}{\sqrt{k}}} = \frac{1}{2} > 0$$

$\sum_{k=1}^{\infty} \frac{1}{k}$ ÄR DIV. TY $\int_1^{\infty} \frac{1}{x}dx$ ÄR DIV ENL. INT. KRIT.

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2k-\sqrt{k}}$ ÄR DIV. ENL. JÄMFÖRELSEKRIT.

<u>SVAR</u>! DIVERGENT

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4. $F(x,y,z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0 \qquad P_0 = (0,4,\frac{3}{5})$

$$\text{GRAD } F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(\frac{2x}{9}, \frac{2y}{25}, 2z\right)$$

$$\bar{n} = \text{GRAD } F_{(0,4,\frac{3}{5})} = \left(0, \frac{8}{25}, \frac{6}{5}\right)$$

π! $\bar{n}\cdot\left((x,y,z)-(0,4,\frac{3}{5})\right) = 0$

$$\left(0, \frac{8}{25}, \frac{6}{5}\right)\left(x-0, y-4, z-\frac{3}{5}\right) = 0$$

$$\frac{8}{25}(y-4) + \frac{6}{5}\left(z-\frac{3}{5}\right) = 0 \Leftrightarrow \frac{8}{25}y - \frac{32}{25} + \frac{6z}{5} - \frac{18}{25} = 0$$

$$\Leftrightarrow 8y + 30z - 50 = 0 \Leftrightarrow 4y + 15z - 25 = 0$$

EFTERSOM PLANET INTE INNEHÅLLER X ÄR DET PARALLELLT MED X-AXELN.

<u>SVAR</u>! π! $4y + 15z - 25 = 0$

// MED X-AXELN.

5. $z = e^{-(x^2+y^2)/8} - e^{-2}$

I xy-PLANET ÄR $z=0$.

$e^{-(x^2+y^2)/8} - e^{-2} = 0 \iff e^{-(x^2+y^2)/8} = e^{-2}$

$-(x^2+y^2)/8 = -2 \iff x^2+y^2 = 16$ CIRKEL $r=4$.

$$V = \iint_D \left(e^{-(x^2+y^2)/8} - e^{-2}\right)dxdy = \left| \begin{matrix} \text{pol. koord} \\ dxdy = r\,drd\theta \\ 0 \le r \le 4 \\ 0 \le \theta \le 2\pi \end{matrix} \right| =$$

$$= \int_0^{2\pi}\int_0^4 \left(e^{-r^2/8} - e^{-2}\right) r\,drd\theta = \int_0^{2\pi} d\theta \int_0^4 \left(re^{-r^2/8} - re^{-2}\right)dr =$$

$$= 2\pi \cdot \left[-4e^{-r^2/8} - \frac{r^2}{2}e^{-2}\right]_0^4 = 2\pi\left(-4e^{-2} - 8e^{-2} - (-4)\right) =$$

$$= 2\pi(4 - 12e^{-2}) = 8\pi(1-3e^{-2}).$$ SVAR! $8\pi(1-3e^{-2})$ v.e.

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6. $f(x,y,z) = (2x-y)e^{-xyz}$ $P_0 = (2,0,1)$ $\bar{v}_1 = (0,0,0)-(2,0,1) = -(2,0,1)$

$|\bar{v}_1| = \sqrt{5}$

I!

$$\frac{df}{d\bar{v}} = \text{grad } f_{P_0} \cdot \frac{\bar{v}}{|\bar{v}|}.$$

$$\text{grad } f = \left(2e^{-xyz} - yz(2x-y)e^{-xyz},\ -e^{-xyz} - \cancel{xz}(2x-y)e^{-xyz},\ -xy(2x-y)e^{-xyz}\right)$$

$$\text{grad} f_{P_0} = (2-0,\ -1-8,\ 0) = (2,-9,0)$$

$$\frac{df}{d\bar{v}_1} = (2,-9,0)\cdot\frac{-1}{\sqrt{5}}(2,0,1) = \frac{-1}{\sqrt{5}}(4) = -\frac{4}{\sqrt{5}} = -\frac{4}{5}\sqrt{5}$$

II! $P_1 = (0,0,0)$ MAX. $\frac{df}{d\bar{v}}$ ÄR I GRADIENTENS RIKTNING.

$$\frac{df}{d\bar{v}}P_1 = (2,-1,0) = \bar{v}_2 \qquad |\bar{v}_2| = \sqrt{5}$$

$$\frac{df}{d\bar{v}}_{max} = (2,-1,0)\cdot\frac{1}{\sqrt{5}}(2,-1,0) = \frac{5}{\sqrt{5}} = \sqrt{5}$$

SVAR: $\frac{df}{d\bar{v}} = -\frac{4}{5}\sqrt{5}$, MAX DERIVATA I ORIGO $= \sqrt{5}$.

7. $f(x,y) = x^3 + 3xy^2 - 15x - 12y$

$f'_x = 3x^2 + 3y^2 - 15 = 3(x^2+y^2-5)$

$f'_y = 6xy - 12 = 6(xy-2)$

$\left.\begin{array}{l} f'_x = 0 \Rightarrow x^2+y^2=5 \\ f'_y = 0 \Rightarrow xy = 2 \end{array}\right\}\begin{array}{l} \text{I} \\ \text{II} \end{array}$ II GER $x = \frac{2}{y}$, $y \neq 0$

$x = \frac{2}{y}$ INSATT I I GER $\left(\frac{2}{y}\right)^2 + y^2 = 5 \Leftrightarrow \frac{4}{y^2} + y^2 = 5$

$\Leftrightarrow 4 + y^4 = 5y^2 \Leftrightarrow y^4 - 5y^2 + 4 = 0 \Leftrightarrow (y^2-1)(y^2-4) = 0$

$y = \pm 1$, $y = \pm 2$ $y = \pm 1 \Rightarrow x = \pm 2$, $y = \pm 2 \Rightarrow x = \pm 1$

STAT. PUNKTER: $(1,2)$, $(-1,-2)$, $(2,1)$, $(-2,-1)$

$f''_{xx} = 6x = A$

$f''_{xy} = 6y = B$

$f''_{yy} = 6x = C$

PUNKT	A	B	C	$AC-B^2$	KAR
$(1,2)$	6	12	6	$6^2-12^2<0$	SADEL
$(-1,-2)$	-6	-12	-6	$(-6)^2-12^2<0$	SADEL
$(2,1)$	12	6	12	$12^2-6^2>0$	LOK.MIN
$(-2,-1)$	-12	-6	-12	$(-12)^2-(-6)^2>0$	LOK MAX

SVAR: $f(1,2) = -26$ o $f(-1,-2) = 26$ ÄR SADELPUNKTER

$f(2,1) = -28$ ÄR LOK. MIN $f(-2,-1) = 28$ ÄR LOK. MAX

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3. $\displaystyle\int_C \frac{x\,dx + (y-1)\,dy}{x^2+y^2-2y+1} = \int_C \frac{x\,dx + (y-1)\,dy}{x^2+(y-1)^2}$ $C: y = \sin x$, $(0,0) \to (2\pi, 0)$

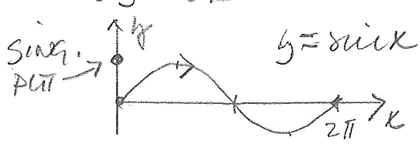
$P = \frac{x}{x^2+(y-1)^2}$ $\frac{\partial P}{\partial y} = \frac{-2x(y-1)}{(x^2+(y-1)^2)^2}$

$Q = \frac{y-1}{x^2+(y-1)^2}$ $\frac{\partial Q}{\partial x} = \frac{-2x(y-1)}{(x^2+(y-1)^2)^2}$

LIKA

SING. PUNKT I $(0,1)$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow$ VI KAN BYTA VÄG. NYA VÄGEN FÅR INTE GÅ GENOM $(0,1)$. TAG T.EX. $x = t$ OCH $y = 0$

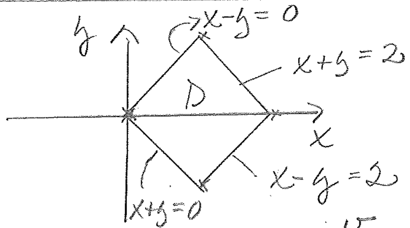


$\displaystyle\int_C = \int_0^{2\pi} \frac{t\,dt}{t^2+1} = \left[\frac{1}{2}\ln(t^2+1)\right]_0^{2\pi} =$

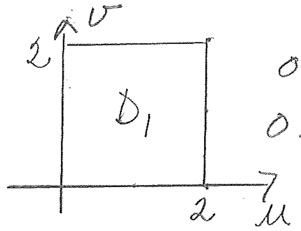
$= \frac{1}{2}\ln(4\pi^2+1) - 0 = \frac{1}{2}\ln(4\pi^2+1)$ SVAR: $\frac{1}{2}\ln(4\pi^2+1)$

9.

$$\iint_D \sqrt{\frac{x-y}{x+y+1}}\,dxdy$$



SÄTT $\begin{cases} x-y=u \\ x+y=v \end{cases} \Leftrightarrow \begin{aligned} x &= \frac{u+v}{2} \\ y &= \frac{v-u}{2} \end{aligned}$



$0 \le u \le 2$
$0 \le v \le 2$

$$\frac{d(x,y)}{d(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4}+\frac{1}{4} = \frac{1}{2} \Rightarrow dxdy = \frac{1}{2}dudv$$

$$\iint_D \sqrt{\frac{x-y}{x+y+1}}\,dxdy = \iint_{D_1} \sqrt{\frac{u}{v+1}}\cdot\frac{1}{2}\,dudv = \frac{1}{2}\int_0^2\int_0^2 \frac{\sqrt{u}}{\sqrt{v+1}}\,dudv$$

$$= \frac{1}{2}\int_0^2 \frac{1}{\sqrt{v+1}}\,dv \int_0^2 \sqrt{u}\,du = \frac{1}{2}\left[\frac{(v+1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_0^2 \cdot \left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^2 =$$

$$= \frac{1}{2}\left[2\sqrt{v+1}\right]_0^2 \cdot \left[\frac{2}{3}u\sqrt{u}\right]_0^2 = \frac{1}{2}\cdot 2(\sqrt{3}-1)\cdot\frac{2}{3}\cdot 2\sqrt{2} =$$

$$= \frac{4}{3}\sqrt{2}(\sqrt{3}-1).$$

SVAR: $\frac{4}{3}\sqrt{2}(\sqrt{3}-1)$

b) OMRÅDET D HAR AREAN $\sqrt{2}\cdot\sqrt{2} = 2$
OMRÅDET D_1 HAR AREAN $2\cdot 2 = 4$
DVS DET NYA OMRÅDET ÄR DUBBELT SÅ STORT.
$\Rightarrow dxdy = \frac{1}{2}dudv.$

SVAR: INTEGRATIONSOMRÅDETS AREA FÖRÄNDRAS, SÅ ATT DESS AREA BLIR DUBBELT SÅ STOR.

10. $F(x, y, z(x,y)) = z e^x + y \sin z - \frac{\pi}{2} - 1 \qquad P = (0, 1, \frac{\pi}{2})$

I: VISA $F(0,1,\frac{\pi}{2}) = 0$ OCH $F'_z(0,1,\frac{\pi}{2}) \neq 0$

$F(0,1,\frac{\pi}{2}) = \frac{\pi}{2} \cdot e^0 + 1 \cdot \sin\frac{\pi}{2} - \frac{\pi}{2} - 1 = \frac{\pi}{2} + 1 - \frac{\pi}{2} - 1$ ok.

$F'_z = e^x + y\cos z \qquad F'(0,1,\frac{\pi}{2}) = e^0 + 1 \cdot \cos\frac{\pi}{2} = 1 \neq 0$ ok.

II BESTÄM $z''_{xy}(0,1)$

$z'_x = \frac{\partial z}{\partial x}$: $\quad z'_x \cdot e^x + z \cdot e^x + y \cdot z'_x \cos z = 0$

$$z'_x(e^x + y\cos z) = -z e^x$$

$$z'_x(0,1) = \frac{-\frac{\pi}{2}}{1} = -\frac{\pi}{2}$$

$z'_y = \frac{\partial z}{\partial y}$: $\quad z'_y e^x + \sin z + y \cdot z'_y \cdot \cos z = 0$

$$z'_y(e^x + y\cos z) = -\sin z$$

$$z'_y(0,1) = \frac{-1}{1} = -1$$

$z''_{xy} = \frac{\partial^2 z}{\partial x \partial y}$: $\quad z''_{xy} e^x + z'_y \cdot e^x + z'_x \cdot \cos z +$
$+ y(z'_{xy} \cos z - z'_y \cdot z'_x \sin z) = 0$

$$z''_{xy}(e^x + y\cos z) = y \cdot z'_y \cdot z'_x \sin z - z'_y e^x - z'_x \cos z$$

$$z''_{xy} = \frac{y \cdot z'_y \cdot z'_x \sin z - z'_y e^x - z'_x \cos z}{e^x + y\cos z}$$

$$z''_{xy}(0,1) = \frac{1 \cdot -1 \cdot -\frac{\pi}{2} \cdot \sin\frac{\pi}{2} - (-1 \cdot e^0) - (-\frac{\pi}{2} \cdot \cos\frac{\pi}{2})}{1}$$

$$z''_{xy}(0,1) = \frac{\frac{\pi}{2} + 1 + 0}{1} = 1 + \frac{\pi}{2}$$

SVAR: $1 + \frac{\pi}{2}$

11.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ SÖK EGENVÄRDENA λ_1 o λ_2

a)

KAR. EKV $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$$(a-\lambda)(d-\lambda) - bc = 0 \Leftrightarrow \lambda^2 - \lambda(a+d) + ad - bc = 0$$

$$\lambda = \frac{a+d}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - ad + bc}$$

$$\lambda = \frac{a+d}{2} \pm \sqrt{\frac{a^2 + 2ad + d^2 - 4ad + 4bc}{4}}$$

$$\lambda = \frac{a+d}{2} \pm \sqrt{\frac{a^2 - 2ab + d^2 + 4bc}{4}} = \frac{a+d}{2} \pm \sqrt{\frac{(a-d)^2 + 4bc}{4}}$$

λ HAR TVÅ OLIKA REELLA VÄRDEN OM

$(a-d)^2 + 4bc > 0$ SVAR: $(a-d)^2 + 4bc > 0$

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b) $\lambda_1 = \frac{1}{2}\left(a+d + \sqrt{(a-d)^2 + 4bc}\right)$ $\lambda_2 = \frac{1}{2}\left(a+d - \sqrt{(a-d)^2 + 4bc}\right)$

$$\lambda_1 \cdot \lambda_2 = \frac{1}{4}\left((a+d)^2 - \left(\sqrt{(a-d)^2 + 4bc}\right)^2\right) =$$

$$= \frac{1}{4}\left((a+d)^2 - \left((a-d)^2 + 4bc\right)\right) =$$

$$= \frac{1}{4}\left(a^2 + 2ad + d^2 - (a^2 - 2ad + d^2 + 4bc)\right) =$$

$$= \frac{1}{4}(4ad - 4bc) = ad - bc.$$

$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ $\therefore \det A = \lambda_1 \cdot \lambda_2$

DÅ λ_1 o λ_2 OLIKA
RELLA TAL.

MATEMATIK II FÖR CL, SF1613. 2010-05-26

11c) ANTAG A ÄR EN $n \times n$ MATRIS

A. ÄR DIAGONALISERBAR $\Leftrightarrow$

1) A HAR n ST LIN. OBEROENDE EGENVEKTORER

2) OM A HAR n ST OLIKA EGENVÄRDEN

3) OM A ÄR SYMMETRISK $\Leftrightarrow$ A ÄR ORTOGONALT DIAGONALISERBAR