FÖRSLAG TILL LÖSNING TENTAMEN I MATEMATIK II FÖR CL 3F1623. 2011-05-25.

1.

$$\begin{cases} x - y + 2z = b \\ x + 6y - 5z = -4 \\ 3x + 4y + az = 2 \end{cases} \quad \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 1 & 6 & -5 \\ 3 & 4 & a \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ -4 \\ 2 \end{pmatrix} \Leftrightarrow A\bar{x} = \bar{b}$$

$$\det A = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 6 & -5 \\ 3 & 4 & a \end{vmatrix} = 6a + 15 + 8 - 36 + 20 + a = 7a + 7$$

FÖR ATT DET SKALL FINNAS OÄNDLIGT MÅNGA LÖSNINGAR MÅSTE $\det A = 0 \Rightarrow 7a + 7 = 0 \Leftrightarrow a = -1$.
INSATT GER DET EKV. SYSTEMET:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & b \\ 1 & 6 & -5 & -4 \\ 3 & 4 & -1 & 2 \end{array}\right) \begin{matrix} (-1) & (-3) \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & b \\ 0 & 7 & -7 & -4-b \\ 0 & 7 & -7 & 2-3b \end{array}\right) (-1) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & b \\ 0 & 7 & -7 & -4-b \\ 0 & 0 & 0 & 6-2b \end{array}\right)$$

OM $6 - 2b = 0$ FINNS OÄNDL. MÅNGA LÖSN. DVS DÅ $b = 3$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 7 & -7 & -7 \end{array}\right) \left(\tfrac{1}{7}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -1 \end{array}\right) (1) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

$$\begin{matrix} x = 2 - z \\ y = -1 + z \\ z = z \end{matrix} \qquad \begin{cases} x = 2 - t \\ y = -1 + t \\ z = t \end{cases}$$

<u>SVAR:</u> $a = -1$, $b = 3$
$$\begin{cases} x = 2 - t \\ y = -1 + t \\ z = t \end{cases}$$

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2. $\bar{v}_1 = (-1, -1, 0, 1)$ $\bar{v}_2 = (1, 2, 3, -1)$, $\bar{v}_3 = (7, 0, 4, 6)$
$\bar{v}_4 = (-6, 0, 3, 3)$

* $a\bar{v}_1 + b\bar{v}_2 + c\bar{v}_3 + d\bar{v}_4 = \bar{0}$
LINJÄRT OBEROENDE ENDAST DÅ $a = b = c = d = 0$

* ÄR ETT HOMOGENT EKV. STUDERA DESS DETERMINANT. OM $\det \neq 0 \Rightarrow$ ENDAST TRIVIALA LÖSN.

$$\begin{vmatrix} -1 & 1 & 7 & -6 \\ -1 & 2 & 0 & 0 \\ 0 & 3 & 4 & 3 \\ 1 & -1 & 6 & 3 \end{vmatrix} \overset{(2)}{=} \begin{vmatrix} -1 & -1 & 7 & -6 \\ -1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 3 \\ 3 & 1 & 6 & 3 \end{vmatrix} = \{\text{UTV. LÄNGS 2:A RADEN}\} =$$

$$= (-1)^{2+1} \cdot (-1) \cdot \begin{vmatrix} -1 & 7 & -6 \\ 3 & 4 & 3 \\ 1 & 6 & 3 \end{vmatrix} \begin{matrix} (3) & (1) \end{matrix} = \begin{vmatrix} -1 & 7 & -6 \\ 0 & 25 & -15 \\ 0 & 13 & -3 \end{vmatrix} = \{\text{UTV. LÄNGS 1:A KOL}\} =$$

$$= (-1)^{1+1} \cdot (-1) \begin{vmatrix} 25 & -15 \\ 13 & -3 \end{vmatrix} = -5 \begin{vmatrix} 5 & -3 \\ 13 & -3 \end{vmatrix} = -5(-15 + 39) = -120 \neq 0$$

$\therefore$ <u>LINJÄRT OBEROENDE.</u>

3. $$\sum_{n=0}^{\infty} \frac{x^n e^n}{n+1} = \sum_{n=0}^{\infty} a_n x^n \qquad a_n = \frac{e^n}{n+1}$$

SÖK KONVERGENSRADIEN R!

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{\frac{e^n}{n+1}}{\frac{e^{n+1}}{n+2}} = \lim_{n\to\infty} \frac{e^n}{n+1} \cdot \frac{n+2}{e^n \cdot e} =$$

$$= \lim_{n\to\infty} \frac{1}{e} \frac{n+2}{n+1} = \frac{1}{e} \quad \therefore \text{ KONV. FÖR } |x| < \frac{1}{e}$$

$$x = \frac{1}{e} \Rightarrow \sum_{n=0}^{\infty} \frac{\left(\frac{1}{e}\right)^n \cdot e^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \text{ DIV. TY } \int_0^{\infty} \frac{1}{x+1} dx \text{ div.}$$

$$x = -\frac{1}{e} \Rightarrow \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{e}\right)^n \cdot e^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \quad \text{1) ALTERNERANDE}$$

2) $$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

3) AVTAGANDE TY $f(x) = \frac{1}{x+1}$ $f'(x) = -\frac{1}{(x+1)^2} < 0$

$\therefore$ KONV. ENL LEIBNIZ

<u>SVAR:</u> KONV FÖR $-\frac{1}{e} \le x < \frac{1}{e}$

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4. $$z = \frac{1}{x} + \frac{1}{y} + xy \qquad x \neq 0, \; y \neq 0$$

$$z'_x = -\frac{1}{x^2} + y \qquad z'_x = 0 \Rightarrow y = \frac{1}{x^2}$$

$$z'_y = -\frac{1}{y^2} + x \qquad z'_y = 0 \Rightarrow x = \frac{1}{y^2}$$

$$y = \frac{1}{\left(\frac{1}{y^2}\right)^2} \Leftrightarrow y = y^4 \qquad y(1-y^3) = 0 \qquad \underline{y = 1}$$

$y = 1 \Rightarrow x = 1 \quad \therefore$ STAT. PUNKT I $(1,1)$

$$z''_{xx} = \frac{2}{x^3} \qquad z''_{xy} = 1 \qquad z''_{yy} = \frac{2}{y^3}$$

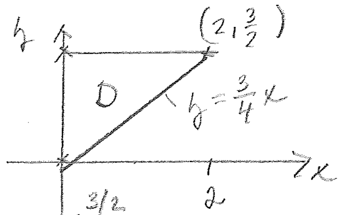
$A = z''_{xx}(1,1) = 2 = z''_{yy}(1,1) = C \qquad z''_{xy} = 1 = B$

$AC - B^2 = 2 \cdot 2 - 1^2 = 3 > 0 \qquad A > 0 \qquad \therefore$ LOK. MIN I $(1,1)$

<u>SVAR:</u> PÖLEN FINNS I $(1,1,3)$

5.

$$\iint_D \sqrt{y^2+4}\,dx\,dy =$$



$$= \int_0^{\frac{3}{2}} \left(\int_0^{\frac{4y}{3}} \sqrt{y^2+4} \right) dx\,dy = \int_0^{3/2} \left[x\sqrt{y^2+4} \right]_0^{\frac{4y}{3}} dy =$$

$$= \int_0^{3/2} \frac{4y}{3}\sqrt{y^2+4}\,dy = \left| \begin{array}{l} y^2+4=t \\ 2y\,dy=dt \\ y=0 \Rightarrow t=4 \\ y=\frac{3}{2} \Rightarrow t=\frac{25}{4} \end{array} \right| = \int_4^{\frac{25}{4}} \frac{2}{3}\sqrt{t}\,dt = \left[\frac{2}{3}\cdot\frac{2}{3}t^{3/2} \right]_4^{\frac{25}{4}}$$

$$= \frac{4}{9}\left(\frac{25}{4}\cdot\frac{5}{2} - 4\cdot 2 \right) = \frac{4}{9}\left(\frac{125}{8} - 8 \right) = \frac{4}{9}\left(\frac{125-64}{8} \right) = \frac{4}{9}\cdot\frac{61}{8} = \frac{61}{18}$$

SVAR! $\frac{61}{18}$

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6. F: $z = xy \Leftrightarrow xy - z = 0$

π: $x + y + z = 0$

GRAD $F = \bar{n}_F = (y, x, -1)$ $\quad \bar{n}_\pi = (1,1,1)$

$\bar{n}_F \parallel \bar{n}_\pi \Leftrightarrow (y, x, -1) = k(1,1,1)$

$$\begin{cases} y = k \\ x = k \\ -1 = k \end{cases} \qquad k=-1 \Rightarrow x = y = -1. \qquad z(1,1) = (-1)\cdot(-1) = 1.$$

∴ PUNKTEN ÄR $(-1,-1,1)$ SVAR! $(-1,-1,1)$

7.

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & -2 \end{pmatrix}.$$

SÖK EGENVÄRDEN!

KAR. EKV.

$$\begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & -2-\lambda \end{vmatrix} = 0 \Leftrightarrow (3-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\big((1-\lambda)(-2-\lambda) - 4\big) = 0 \Leftrightarrow (3-\lambda)(\lambda^2 + \lambda - 6) = 0$$

$$(3-\lambda)(\lambda+3)(\lambda-2) = 0 \Rightarrow \lambda_1 = -3,\ \lambda_2 = 2,\ \lambda_3 = 3.$$

FORTS.

7a) FORTS. SÖK EGENVEKTORER:

$\lambda_1 = -3$: $\left(\begin{array}{ccc|c} 6 & 1 & 2 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 6 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) (-2) \sim \left(\begin{array}{ccc|c} 6 & -3 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$

$2y + z = 0 \Leftrightarrow z = -2y$

$6x - 3y = 0 \Leftrightarrow x = \frac{3y}{6}$

$\begin{cases} x = \frac{1}{2}t \\ y = t \\ z = -2t \end{cases}$ $\bar{v}_1 = t\begin{pmatrix} \frac{1}{2} \\ 1 \\ -2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ $t \neq 0$

$\lambda_2 = 2$: $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) (1) \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$

$y - 2z = 0 \Leftrightarrow y = 2z$

$x + 2y = 0$ $\quad x = -2y = -2 \cdot 2z$

$\begin{cases} x = -4t \\ y = 2t \\ z = t \end{cases}$ $\bar{v}_2 = t\begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}, t \neq 0$

$\lambda = 3$ $\left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -5 & 0 \end{array}\right) (1) (-1) \sim \left(\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right)$ $\begin{matrix} x = t \\ y = 0 \\ z = 0 \end{matrix}$ $\bar{v}_3 = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $t \neq 0$

<u>SVAR</u>: $\lambda_1 = -3$ $\bar{v}_1 = t\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

$\lambda_2 = 2$ $\bar{v}_2 = t\begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$

$\lambda_3 = 3$ $\bar{v}_3 = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $t \neq 0$

b)

$C = \begin{pmatrix} b & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 6 & 4 \end{pmatrix}$ FÖR ATT C SKALL HA INVERS MÅSTE DESS DETERMINANT $\neq 0$.

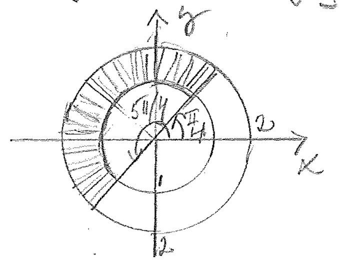
$\begin{vmatrix} b & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 6 & 4 \end{vmatrix} = b \cdot \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = b \cdot 0 = 0$

det $C = 0$ OBEROENDE AV b. ∴ DET FINNS INGET VÄRDE PÅ b SOM GÖR ATT C KAN VARA INVERS TILL A.

<u>SVAR</u>! C KAN ALDRIG BLI INVERS TILL A.

8.

$$\iint_D \frac{x^2}{1+(x^2+y^2)^2}\,dxdy \qquad D=\{(x,y): 1\le x^2+y^2\le 4 \quad x\le y\}$$



POLÄRA KOORD.
$x = r\cos\theta$
$y = r\sin\theta$
$dxdy = r\,dr\,d\theta$
$1 \le r \le 2$
$\frac{\pi}{4} \le \theta \le \frac{5\pi}{4}$

$$= \int_{\pi/4}^{5\pi/4}\int_1^2 \frac{r^2\cos^2\theta}{1+(r^2)^2}\cdot r\,dr\,d\theta$$

$$= \int_{\pi/4}^{5\pi/4}\cos^2\theta\,d\theta \int_1^2 \frac{r^3}{1+r^4}\,dr = \int_{\pi/4}^{5\pi/4}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \cdot \left[\frac{1}{4}\ln(1+r^4)\right]_1^2 =$$

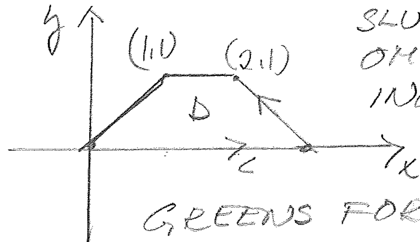
$$= \left[\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_{\pi/4}^{5\pi/4} \cdot \left(\frac{1}{4}\ln 17 - \frac{1}{4}\ln 2\right) =$$

$$= \frac{1}{2}\left(\frac{5\pi}{4} + \underbrace{\frac{\sin\frac{5\pi}{2}}{2}}_{\frac{1}{2}} - \frac{\pi}{4} - \underbrace{\frac{\sin\frac{\pi}{2}}{2}}_{\frac{1}{2}}\right)\cdot\frac{1}{4}\ln\frac{17}{2} = \frac{1}{2}\cdot\pi\cdot\frac{1}{4}\ln\frac{17}{2}$$

$$= \frac{\pi}{8}\ln\frac{17}{2}.$$ <u>SVAR</u>! $\frac{\pi}{8}\ln\frac{17}{2}$

9.

$$\int_C (\sin^2 x - y\cos^2 x)\,dx + \left(\frac{x}{2} - \frac{1}{4}\sin 2x\right)dy$$



SLUTET OMRÅDE. INGA SING. PUNKTER

GREENS FORMEL KAN ANVÄNDAS

$P = \sin^2 x - y\cos^2 x$

$Q = \frac{x}{2} - \frac{1}{4}\sin 2x$

$$\frac{\partial P}{\partial y} = -\cos^2 x \qquad \frac{\partial Q}{\partial x} = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2}(1-\cos 2x) = \sin^2 x$$

$$\int_C = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_D \sin^2 x - (-\cos^2 x)\,dxdy =$$

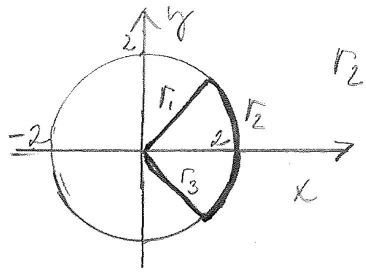
$$= \iint_D \sin^2 x + \cos^2 x\,dxdy = \iint_D dxdy = \left|\begin{matrix}\text{AREAN AV} \\ \text{PARALLELLTRAPETSET}\end{matrix}\right|$$

$$= \frac{1}{2}(1+3) = 2.$$

<u>SVAR</u>! 2

10. $T(x,y) = x^2 - xy + y^2$

SÖK STÖRSTA OCH MINSTA VÄRDE PÅ RANDEN.



$r_2: x^2+y^2=4$

SKÄRNINGSPUNKTER:

MELLAN r_1 OCH r_2 $\begin{cases} y = x \\ x^2+y^2=4 \end{cases} \Rightarrow 2x^2 = 4$, $x = (\pm)\sqrt{2}$

MELLAN r_2 OCH r_3 $\begin{cases} x^2+y^2=4 \\ y=-x \end{cases} \Rightarrow x = \sqrt{2}$

TEMPERATUREN PÅ:

RANDEN r_1: $y = x$ $\quad 0 \le x \le \sqrt{2}$

$T(x,x) = x^2 - x^2 + x^2 = x^2 = f(x) \qquad f'(x) = 2x$

$f'(x) = 0 \Rightarrow x = 0 \qquad f(0) = \underline{\underline{0}}$

RANDEN r_2: $x^2+y^2=4 \qquad x = \sqrt{4-y^2} \quad \sqrt{2} \le x \le 2, \ -\sqrt{2} \le y \le \sqrt{2}$

$T(\sqrt{4-y^2}, y) = 4-y^2 - \sqrt{4-y^2}\cdot y + y^2 = 4 - y\sqrt{4-y^2} = g(y)$

$g'(y) = -\sqrt{4-y^2} - y\cdot\frac{1}{2}\frac{-2y}{\sqrt{4-y^2}} = -\sqrt{4-y^2} + \frac{y^2}{\sqrt{4-y^2}} =$

$= \frac{-(4-y^2)+y^2}{\sqrt{4-y^2}} = \frac{-4+2y^2}{\sqrt{4-y^2}} \qquad g'(y) = 0 \Rightarrow y = \pm\sqrt{2}$

$g(\sqrt{2}) = 4 - \sqrt{2}\cdot\sqrt{4-2} = 4-2 = \underline{\underline{2}} \qquad g(-\sqrt{2}) = 4 + \sqrt{2}\cdot\sqrt{2} = \underline{\underline{6}}$

RANDEN r_3: $y = -x \qquad 0 \le x \le \sqrt{2}$

$T(x,-x) = x^2 + x^2 + x^2 = 3x^2 = h(x) \qquad h'(x) = 6x$

$h'(x) = 0 \Rightarrow x = 0 \qquad h(0) = \underline{\underline{0}}$

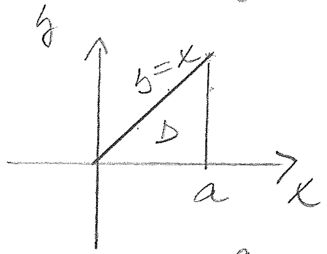
HÖRN: $(0,0)$, $(\sqrt{2}, \sqrt{2})$ OCH $(\sqrt{2}, -\sqrt{2})$

$T(0,0) = 0 \qquad T(\sqrt{2},\sqrt{2}) = 2-2+2 = \underline{\underline{2}} \qquad T(\sqrt{2},-\sqrt{2}) = 2+2+2 = \underline{\underline{6}}$

STÖRSTA VÄRDE: 6
MINSTA VÄRDE: 0

$\therefore$ SVAR: HÖGSTA TEMP ÄR 6
MINSTA TEMP ÄR 0

11. $F(a)=\int_0^a \frac{\sin ax}{x}\,dx \qquad G(a)=2\int_0^a \frac{\sin x^2}{x}\,dx$



$0\le y\le x\le a$

$I_1=\iint_D \cos xy\,dx\,dy=$

$=\int_0^a\int_y^a \cos xy\,dx\,dy=\int_0^a\left[\frac{\sin xy}{y}\right]_y^a dy=$

$=\int_0^a\left(\frac{\sin ay}{y}-\frac{\sin y^2}{y}\right)dy=\left|\begin{matrix} y=x \\ dy=dx \\ y=0\Rightarrow x=0 \\ y=a\Rightarrow x=a\end{matrix}\right|=$

$=\int_0^a \frac{\sin ax}{x}\,dx-\int_0^a\frac{\sin x^2}{x}\,dx = F(a)-\frac{G(a)}{2}$

$I_2=\iint_D \cos xy\,dxy=\int_0^a\int_0^x \cos xy\,dy\,dx=$

$=\int_0^a\left[\frac{\sin xy}{x}\right]_0^x dx=\int_0^a\frac{\sin x^2}{x}\,dx=\frac{G(a)}{2}$

$I_1=I_2 \quad\Rightarrow\quad F(a)-\frac{G(a)}{2}=\frac{G(a)}{2}$

$F(a)=\frac{G(a)}{2}+\frac{G(a)}{2}=G(a)$ v.s.v.