FÖRSLAG TILL LÖSNING TENTAMEN I MATEMATIK II 1(7)
FÖR CL. SF 1613. 2010-08-24

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1. $X = A^{-1} \cdot B^{-1} = (BA)^{-1}$ $\quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$BA = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(BA)^{-1}: \quad \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \updownarrow \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \textcircled{-2}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \textcircled{-1}\ \textcircled{1} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underbrace{\begin{matrix} -2 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{matrix}}_{(BA)^{-1}}$$

SVAR: $X = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

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2. a) $A = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$ KAR. EKV.: $\begin{vmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} = 0$

$\Leftrightarrow (4-\lambda)^2 = 0$ $\quad \lambda = 4$ (DUBBELROT) ÄR EGENVÄRDEN

$\lambda = 4 \Rightarrow \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad \begin{matrix} y = 0 \\ x = t \end{matrix}$ $\quad \bar{v} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad t \neq 0$ ÄR EGENVEKTO

b) $A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ KAR. EKV.: $\begin{vmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = 0$

$\Leftrightarrow (4-\lambda)^2 = 0$ $\quad \lambda = 4$ ÄR EGENVÄRDEN

$\lambda = 4 \Rightarrow \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad \begin{cases} x = s \\ y = t \end{cases}$ $\quad \bar{v} = s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad s, t \neq 0$

SVAR: a) EGENVÄRDEN: $\lambda = 4$, EGENVEKTOR: $\bar{v} = t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) EGENVÄRDEN: $\lambda = 4$, EGENVEKTOR: $\bar{v} = s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$s, t \neq 0$

3a) $f(x,y)=\sqrt{\frac{xy}{x^2+y^2}}$

$\frac{xy}{x^2+y^2} \geqslant 0 \iff xy \geqslant 0 \iff x \geqslant 0$ OCH $y \geqslant 0$

ELLER $x \leq 0$ OCH $y \leq 0$, DESSUTOM $x \neq 0 \;\underline{o}\; y \neq 0$

TILLSAMMANS! $x<0 \;\underline{o}\; y<0$ ELLER $x>0 \;\underline{o}\; y>0$

b) $$\lim_{(x,y)\to(-1,1)} \frac{x^2-y^2}{x^2-xy-2y^2} = \lim_{(x,y)\to(-1,1)} \frac{(x+y)(x-y)}{x^2-y^2-xy-y^2} =$$

$$= \lim_{(x,y)\to(-1,1)} \frac{(x+y)(x-y)}{(x+y)(x-y)-y(x+y)} = \lim_{(x,y)\to(-1,1)} \frac{(x+y)(x-y)}{(x+y)(x-y-y)}$$

$$= \lim_{(x,y)\to(-1,1)} \frac{x-y}{x-2y} = \frac{-1-1}{-1-2} = \frac{2}{3}$$

<u>SVAR</u>: a) $x<0$ OCH $y<0$ ELLER $x>0$ OCH $y>0$

b) $\frac{2}{3}$.

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4. $F(x,y,z)= x^3+3x^2y^2+y^3+4xy-z^2=0$

$G(x,y,z)= x^2+y^2+z^2-11=0 \qquad P=(1,1,3)$

$\text{GRAD}\,F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(3x^2+6xy^2+4y,\ 6x^2y+3y^2+4x,\ -2z\right)$

$\text{GRAD}\,F(1,1,3) = (13, 13, -6)$

$\text{GRAD}\,G = \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\right) = (2x, 2y, 2z) \qquad \text{GRAD}\,G(1,1,3) = (2,2,6)$
$= 2(1,1,3)$

TANGENTENS RIKTNINGSVEKTOR $\bar{v} = \text{GRAD}\,F \times \text{GRAD}\,G$.

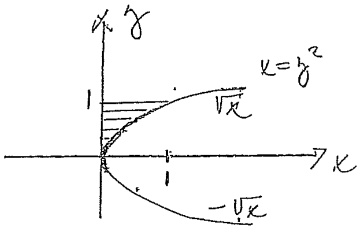
$$\bar{v} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 13 & 13 & -6 \\ 1 & 1 & 3 \end{vmatrix} = (45, -45, 0)$$

TANGENTEN! $\begin{cases} x = 1+45t \\ y = 1-45t \\ z = 3 \end{cases}$

<u>SVAR</u>! $\begin{cases} x = 1+45t \\ y = 1-45t \\ z = 3 \end{cases}$

5.

$$\int_0^1\int_0^{y^2} 3y^3 e^{xy}\,dx\,dy =$$



$x=0$
$x=y^2$

$$= \int_0^1 3y^2 \int_0^{y^2} y e^{xy}\,dx\,dy =$$

$$= \int_0^1 3y^2 \left[e^{xy}\right]_0^{y^2} = \int_0^1 3y^2(e^{y^3}-e^0)\,dy =$$

$$= \int_0^1 (3y^2e^{y^3}-3y^2)\,dy = \left[e^{y^3}-y^3\right]_0^1 = e-1-(1-0)=e-2$$

SVAR: $e-2$, SE FIG. OVAN.

6.

$$f(x,y) = x \arctan\frac{y}{x}$$

$\bar{v} = (-1,0)-(1,-1) = (-2,1)$

$|\bar{v}| = \sqrt{5}$ $\quad \frac{\bar{v}}{|\bar{v}|} = \frac{1}{\sqrt{5}}(-2,1)$

$$\frac{df}{d\bar{v}} = \text{grad } f \cdot \frac{\bar{v}}{|\bar{v}|}$$

$$\text{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\arctan\frac{y}{x} + x\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right), x\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}\right.$$

$$= \left(\arctan\frac{y}{x} - \frac{y}{x}\cdot\frac{1}{1+\frac{y^2}{x^2}},\ \frac{1}{1+\frac{y^2}{x^2}}\right)$$

$$\text{grad} f(1,-1) = \left(\arctan(-1) - \left(\frac{-1}{1}\right)\cdot\frac{1}{1+1},\ \frac{1}{1+1}\right) =$$

$$= \left(-\frac{\pi}{4}+\frac{1}{2},\ \frac{1}{2}\right)$$

$$\frac{df}{d\bar{v}} = \left(-\frac{\pi}{4}+\frac{1}{2},\ \frac{1}{2}\right)\cdot\frac{1}{\sqrt{5}}(-2,1) = \frac{1}{\sqrt{5}}\left(+\frac{\pi}{2}-1+\frac{1}{2}\right) =$$

$$= \frac{1}{\sqrt{5}}\left(\frac{\pi}{2}-\frac{1}{2}\right) = \frac{1}{2\sqrt{5}}(\pi-1)$$

SVAR: $\frac{df}{d\bar{v}} = \frac{1}{2\sqrt{5}}(\pi-1)$

7. $$\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{2n-1}{2}\right)\pi = 1\cdot\sin\frac{\pi}{2} + \frac{1}{2}\sin\frac{3\pi}{2} + \frac{1}{3}\sin\frac{5\pi}{2} + \dots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \dots = \sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}$$ ALTERNERANDE SERIE.

LEIBNIZ KRIT.

1) ALT. SERIE

2) $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n} = 0$

3) AVT. SERIE ? $f(x) = \frac{1}{x}$ $f'(x) = -\frac{1}{x^2} < 0$ ∴ AVTAG FUNKTION $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n}$ ÄR AVTAGANDE

∴ LEIBNIZ KONVERGENT. SVAR: KONVERGENT

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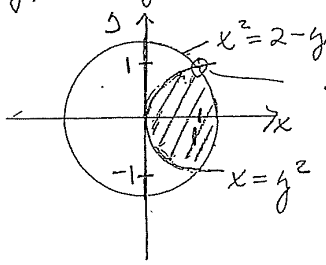
8. $$\iint_D \frac{2x}{1+y^2}\,dx\,dy = \qquad D = \{(x,y): x^2+y^2 \le 2,\ x > y^2\}$$



$x^2 = 2-x$
$x = 1$
$(x = -2)$

$$= \int_{-0}^{1}\int_{y^2}^{\sqrt{2-y^2}} \frac{2x}{1+y^2}\,dx\,dy +$$

$$+ \int_{-1}^{0}\left(\int_{y^2}^{-\sqrt{2-y^2}} \frac{2x}{1+y^2}\right)dx\,dy = \int_0^1 \left[\frac{x^2}{1+y^2}\right]_{y^2}^{\sqrt{2-y^2}} dy +$$

$$+ \int_{-1}^{0}\left[\frac{x^2}{1+y^2}\right]_{y^2}^{-\sqrt{2-y^2}} dy = \int_0^1 \left(\frac{2-y^2-y^4}{1+y^2}\right)dy + \int_{-1}^{0}\left(\frac{2-y^2-y^4}{1+y^2}\right)dy =$$

$$= \int_0^1\left(-y^2 + \frac{2}{1+y^2}\right)dy + \int_{-1}^{0}\left(-y^2 + \frac{2}{1+y^2}\right)dy =$$

$$= \left[-\frac{y^3}{3} + 2\arctan y\right]_0^1 + \left[-\frac{y^3}{3} + 2\arctan y\right]_{-1}^{0} =$$

$$= -\frac{1}{3} + 2\arctan 1 - 0 + 0 - \left(+\frac{1}{3} + 2\arctan(-1)\right) =$$

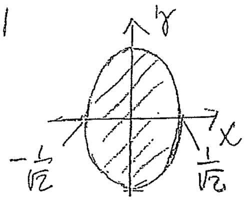
$$= -\frac{1}{3} + \frac{\pi}{2} - \frac{1}{3} + \frac{\pi}{2} = \pi - \frac{2}{3}.$$

SVAR: $\pi - 2/3$

9. $f(x,y) = (3x^2+y^2)e^{2x^2+y^2}$ $\quad 2x^2+y^2 \le 1$

$2x^2+y^2 = 1$ ELLIPS

$\frac{x^2}{(\frac{1}{\sqrt{2}})^2} + y^2 = 1$



STAT. INRE PUNKTER:

$f'_x = (6x + (3x^2+y^2)4x)e^{2x^2+y^2}$

$= 2x(3 + 6x^2 + 2y^2)e^{2x^2+y^2}$

$f'_y = (2y + (3x^2+y^2)\cdot 2y)e^{2x^2+y^2} =$

$= 2y(1 + 3x^2 + y^2)e^{2x^2+y^2}$

$\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$ GER ENDAST (0,0) SOM LÖSNING.

$f(0,0) = 0$

RANDEN: $2x^2+y^2 = 1 \Leftrightarrow y^2 = 1-2x^2$

$f(x, \pm\sqrt{1-2x^2}) = (3x^2+1-2x^2)e^1 = (x^2+1)e = g(x)$

$g'(x) = 2xe \quad g'(x) = 0 \Leftrightarrow x = 0 \quad g(0) = e$

"HÖRN" $(\pm\frac{1}{\sqrt{2}}, 0)$ $\quad f(\pm\frac{1}{\sqrt{2}}; 0) = 3\cdot\frac{1}{2}e^{2\cdot\frac{1}{2}} = \frac{3}{2}e$

STÖRST: $\frac{3}{2}e$, MINST: 0

SVAR! STÖRST VÄRDE: $f(\pm\frac{1}{\sqrt{2}} 0) = \frac{3}{2}e$
MINST VÄRDE: $f(0,0) = 0$

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10. $\int_C \frac{2x\,dx + ay\,dy}{x^2+2y^2+1}$ $\quad P = \frac{2x}{x^2+2y^2+1}$ $\quad Q = \frac{ay}{x^2+2y^2+1}$

$\int_C$ OBEROENDE AV VÄGEN DÅ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial P}{\partial y} = \frac{-2x\cdot 4y}{(x^2+2y^2+1)^2}$ $\quad \frac{\partial Q}{\partial x} = \frac{-2axy}{(x^2+2y^2+1)^2}$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ DÅ $-8 = -2a \Leftrightarrow a = 4$.

FORTS.

10 FORTS. $\int_C \frac{2x\,dx + 4y\,dy}{x^2+2y^2+1}$

$$\frac{\partial F(x,y)}{\partial x} = P = \frac{2x}{x^2+2y^2+1}$$

$$F(x,y) = \ln(x^2+2y^2+1) + g(y)$$

$$\frac{\partial F}{\partial y} = \frac{4y}{x^2+2y^2+1} + g'(y) \equiv Q = \frac{4y}{x^2+2y^2+1}$$

$$\Rightarrow \quad g'(y) = 0 \Leftrightarrow g(y) = C$$

$$\therefore \; F(x,y) = \ln(x^2+2y^2+1) + C$$

$$\int_C = \int_{(1,0)}^{(0,1)} = \left[\ln(x^2+2y^2+1)\right]_{(1,0)}^{(0,1)} = F(0,1) - F(1,0) =$$

$$= \ln 3 - \ln 2 = \ln\frac{3}{2}$$

SVAR! $\ln\frac{3}{2}$

11.a) A SYMMETRISK $n \times n$ MATRIS $\Leftrightarrow$ A DIAGONALISERBAR

DVS $P^{-1}AP = D$ DÄR D ÄR MATRISEN

$$\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & & \lambda_n \end{pmatrix}$$

λ_i ÄR EGENVÄRDEN TILL A OCH P ÄR EN ON-MATRIS BESTÅENDE AV A'S EGENVEKTORER

D ÄR DIAGONAL $\Rightarrow$ $D^2 = \begin{pmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n^2 \end{pmatrix}$ $\lambda_i \geq 0$ $i = 1,2,3$

OCH $\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & & & 0 \\ 0 & \cdots & & \sqrt{\lambda_n} \end{pmatrix}$

$$P^{-1}AP = D \Leftrightarrow A = PDP^{-1}$$

LÅT $B = P\sqrt{D}P^{-1} \Rightarrow B^2 = P\sqrt{D}\underbrace{\boxed{P^{-1}\cdot P}}_{=I}\sqrt{D}P^{-1} = P(\sqrt{D})^2P^{-1}$

$= PDP^{-1} = A$ V.S.V.

11b) $B^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{pmatrix}$ ANVÄND 11a) B^2 SYMM. 3×3 MATRIS.

1) SÖK EGENVÄRDENA TILL B^2.

KAR. EKV: $\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 6-\lambda & 2 \\ 2 & 2 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda^2 + 8 + 8 - $
$-4(6-\lambda) - 4(1-\lambda) - 4(6-\lambda) =$

$= (1-\lambda)(6-\lambda)^2 - 68 + 12\lambda = -\lambda^3 + 13\lambda^2 + 36\lambda = -\lambda(\lambda^2 - 13\lambda + 36) = 0$

$\lambda_1 = 0, \ \lambda_2 = 4, \ \lambda_3 = 9$; $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \Rightarrow \sqrt{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

2) SÖK RESP. EGENVEKTORER TILL B^2.

$\lambda_1 = 0$ $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 6 & 2 & 0 \\ 2 & 2 & 6 & 0 \end{array}\right) \overset{(-2)}{} \sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$ $\begin{cases} x = -4t \\ y = t \\ z = t \end{cases}$

VÄLJ $\bar{v}_1 = (4, -1, -1)$ $|\bar{v}_1| = \sqrt{18}$

$\lambda_2 = 4$ $\left(\begin{array}{ccc|c} -3 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$ $\begin{cases} x = 0 \\ y = t \\ z = -t \end{cases}$

VÄLJ $\bar{v}_2 = (0, 1, -1)$ $|\bar{v}_2| = \sqrt{2}$

$\lambda_3 = 9$ $\left(\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 2 & -3 & 2 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right) \overset{(-1) \ ④}{\sim} \left(\begin{array}{ccc|c} 0 & 10 & -10 & 0 \\ 0 & -5 & 5 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right)$ $\begin{cases} 2x = 3z - 2y = t \\ x = t/2 \\ y = z = t \end{cases}$

$\bar{v}_3 = t_1 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} = t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ VÄLJ $\bar{v}_3 = (1, 2, 2)$ $|\bar{v}_3| = 3$

$P = \begin{pmatrix} \frac{4}{\sqrt{18}} & 0 & \frac{1}{3} \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \\ -\frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} & \frac{2}{3} \end{pmatrix}$ $P^{-1} = P^T =$ (P ON-MATRIS) $\begin{pmatrix} \frac{4}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

$B = P\sqrt{D}P^{-1} =$ (INSATT MATRISER ENL OVAN) $= \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{7}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{pmatrix}$

<u>SVAR</u>: $B = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix}$