SF2715 Applied Combinatorics
Notes and Exercises, Part IV

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This is supplementary material to part IV of SF2715 Applied Combinatorics. First is some theory, including a list of commonly used permutation groups. Then we discuss Burnside's Lemma and the Cycle index theorem, i.e. material Corresponding to chapter 15 in Cameron.

Innehåll

1 Introduction

Let V be a finite set. Most of we will do can be expressed in terms of colorings of the elements in V. that is, functions from V to a given set S. An important special case is if $S = \{0, 1\}$. In that case one may identify a coloring with the subset of V having the "color" 1. We will mostly not have restrictions on the coloring of adjacent elements, but the theory is applicable also for problems with such restrictions.

For a permutation g of the set V and a coloring $c : V \to S$, define $cg = c \circ g$. This is the coloring that gives the color $c \circ g(v)$ to the element v for every $v \in V$. We say that g *acts* on the coloring c by mapping c to cg. Please note that this is the composition of two functions and should be done in that way.

Clearly, cg gives the same color to v as c gives to $g(v)$. Put in other words c gives the same color to v as cg gives to $g^{-1}(v)$.

A set G of permutations of V is a *permutation group* on V if the set is closed under composition and inverse. This is to say that:

- For all $g_1, g_2 \in G$ we have $g_1 g_2 = g_1 \circ g_2 \in G$.
- For all $g \in G$ we have $g^{-1} \in G$.

Let G be a permutation group on V. Let X be a set of colorings with the following property:

- For every $g \in G$ and $c \in X$ we have $cg \in X$.

Then we say that G *acts* on X. Note that for $g, h \in G$ and $c \in X$ that

$$(cg)h = (cg) \circ h = c \circ g \circ h = c \circ (gh) = c(gh).$$

In particular, $(cg)g^{-1} = c$ for all $g \in G$ and $c \in X$.

We consider two colorings c and c' as equivalent if there is a group element g such that $cg = c'$. One may show that this gives an equivalence relation by using the axioms above and the identity $(cg)h = c(gh)$. The equivalence classes are called *orbits* (*banor* in Swedish).

As an example, let X be the set of all colorings with the colors $1, \ldots, r$ of vertices in a regular n-gon. Number the vertices $0, \ldots, n-1$ increasingly in the clockwise direction. Let c be a coloring, and let g be the permutation $(1, 2, 3, \ldots, n)$. By what we discussed before cg is the coloring that gives i the color that $g(i) = (i + 1) \bmod n$ had before. One may image this as if g rotates the n-gon one step clockwise and the colors are still. Alternatively you may image this as the n-gon being still and the colors rotating. Then the colors are rotating "backwards" so the color for vertex i is given to vertex $(i - 1) \bmod n$. The permutation g generates a group with n elements giving a partition of X in orbits. Two colorings are in the same orbit if and only if one can be obtained from the other by rotation.

2 A handful common permutation groups

In this section we give examples of some of the most common permutation groups that you should know about. If nothing else is stated we will express the permutations in cycle format. We will often suppress the commas and write $(a_1a_2\cdots a_k)$ for $(a_1, a_2, \ldots, a_k)$. If a permutation consists of e_i cycles of length i we say that the permutation has cycle structure $1^{e_1}2^{e_2}\cdots n^{e_n}$, where n is the size of V. (In reality we leave out all i such that $e_i = 0$, and we will often write it in reverse order with the longest cycles first.)

The symmetric group S_n, a group with $n!$ elements. This group consists of all permutations of the set $\{1, \ldots, n\}$. Examples:

- S_3. The group consists of the identity (1)(2)(3), the three transpositions (12)(3), (13)(2) och (23)(1) and the two rotations (123) and (132). The identity has cycle structure 1^3 (three 1-cycles), the transpositions have cycle structure $2^1 1^1$ (one 2-cycle and one 1-cycle) and the rotations have cycle structure 3^1 (one 3-cycle).

- S_4. The group consists of 24 elements:

Cycle structure	Number of group elements
1^4	1
$2^1 1^2$	6
2^2	3
$3^1 1^1$	8
4^1	6

  (The number of group elements with a given cycle structure can be computed using Proposition 13.5 in Cameron.)

- S_5. The group consists of 120 elements:

Cycle structure	Number of group elements
1^5	1
$2^1 1^3$	10
$2^2 1^1$	15
$3^1 1^2$	20
$3^1 2^1$	20
$4^1 1^1$	30
5^1	24

The cyclic group C_n, a group with n elements. This group consists of all permutations of vertices of a regular n-gon obtained by rotation. We discussed this group at the end of the Introduction. See Example 2 in Section 15.4 in Cameron for more information. The central property of C_n is that the permutation corresponding to rotation with r steps (that is $360r/n$ degrees) consists of d cycles, each having n/d elements, where d is the greatest common divisor of n and r. Rotation with 9 steps of a 24-gon thus correspond to a permutation with 3 cycles with $24/3 = 8$ elements each, since $\gcd(24, 9) = 3$.

The dihedral group D_n, a group with $2n$ elements. This group consists of all permutations of the vertices in a regular n-gon obtained by rotation and reflection. This is covered in more detail in Example 2 in Section 15.4 in Cameron. Some special cases that are particularly useful are the following.

- D_4. The group acts on a square with vertices 1, 2, 3 and 4, labelled clockwise. The eight group elements are as follows.

(1)(2)(3)(4)	Identity
(1234) (1432)	Rotation $\pm 90°$
(13)(24)	Rotation 180°
(14)(23) (12)(34)	Reflection in a vertical or horizontal line
(13)(2)(4) (24)(1)(3)	Reflection in a diagonal

- D_5. The group acts on a regular pentagon with vertices 1, 2, 3, 4 and 5, labelled clockwise. The ten group elements are as follows.

(1)(2)(3)(4)(5)	Identity
(12345) (15432)	Rotation $\pm 72°$
(13524) (14253)	Rotation $\pm 144°$
(34)(52)(1) (45)(13)(2) (51)(24)(3) (12)(35)(4) (23)(41)(5)	Reflection in line through a vertex and opposite edge

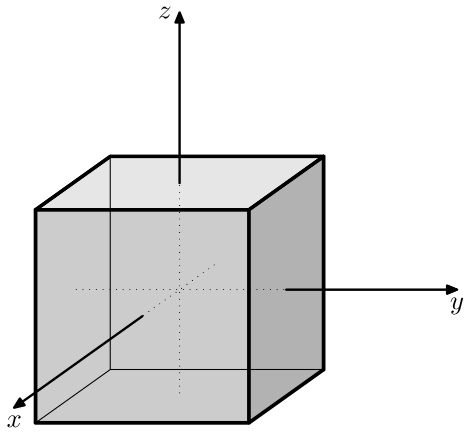
- D_6. The group acts on regular hexagon with vertices 1, 2, 3, 4, 5 and 6, labelled clockwise. The twelve group elements are as follows.

(1)(2)(3)(4)(5)(6)	Identity
(123456) (165432)	Rotation $\pm 60°$
(135)(246) (153)(264)	Rotation $\pm 120°$
(14)(25)(36)	Rotation 180°
(12)(45)(36) (23)(56)(41) (34)(61)(52)	Reflection in a line through two opposite edges
(26)(35)(1)(4) (31)(46)(2)(5) (42)(51)(3)(6)	Reflection in a line through two opposite vertices

Note that D_6 has a more complicated structure than D_5. The reason is that 5 is a prime, while 6 is a composite number.

The group of rotations for the 3-dimensional cube, a group with 24 elements. We get a permutation of the faces of a 3-dimensional cube by rotating the cube. The set of rotations give rise to a permutation group of size 24. This is because any rotation is uniquely determined by which face is the top and which face is the front. kub genom att rotera kuben. There are six choices for the top and for each top face there are four choices for the front, thus a total of $6 \cdot 4 = 24$ possibilities.

In Section 15.2 of Cameron it is described (without any details) how the group elements can be divided into five classes. We may describe these classes explicitly by first place the vertices of the cube in the eight points $\pm e_1 \pm e_2 \pm e_3 = (\pm 1, \pm 1, \pm 1)$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. See Figure 1 for an illustration.



Figur 1: Cube with vertices $(\pm 1, \pm 1, \pm 1)$.

Let us study the cycle structure of the elements in the group. every group element is thus a permutation of a set of six elements (the faces). We get the following table.

Cycle structure	Number of elements	Description
1^6	1	Identity
$2^2 1^2$	3	Rotation 180° around x-, y- or z-axis
$4^1 1^2$	6	Rotation $\pm 90°$ around x-, y- or z-axis
2^3	6	Rotation 180° around the axis $e_i \pm e_j$, $i < j$
3^2	8	Rotation $\pm 120°$ around the axis $e_1 \pm e_2 \pm e_3$

Some comments:

- Note that rotation clockwise around the axis v is equivalent with rotation counter clockwise around the axis $-v$.
- Rotation 180° around an axis of the form $e_i \pm e_j$ means that we rotate a half turn around a line going through two opposite edges. For example in the case when the axis is $e_1 + e_2$ we get a rotation around a line through the edge between the vertices $(e_1 + e_2) + e_3$ and $(e_1 + e_2) - e_3$ and the edge between the vertices $-(e_1 + e_2) + e_3$ and $-(e_1 + e_2) - e_3$.

- Rotation around an axis of the form $v = e_1 \pm e_2 \pm e_3$ means rotation around a line through the vertices v and $-v$.

We could also construct a group consisting of the permutations of the 8 vertices of the cube obtained by rotations. Again we get a group with 24 elements. We have the following table.

Cycle structure	Number of elements	Description
1^8	1	Identity
2^4	3	Rotation 180° around x-, y- r z-axis
4^2	6	Rotation ±90° around x-, y- or z-axis
2^4	6	Rotation 180° around axis $e_i \pm e_j$, $i < j$
$3^2 1^2$	8	Rotation ±120° around axis $e_1 \pm e_2 \pm e_3$

A third possibility is to study the group consisting of the permutations of the 12 edges of the cube obtained by rotations, see Exercise Ö.3.

Product groups. Let G and H be two permutation groups on the sets V and W respectively. We can build the permutation group $G \times H$ on the product set $V \times W$ by defining

$$(g,h)(v,w) = (g(v), h(w)).$$

Note that

$$(g,h)(g',h') = (gg', hh'),$$

and the inverse to (g,h) is equal to (g^{-1}, h^{-1}).

Let us compute the cycle structure for an element (g,h) in a specific example. Let $V = \{1,2\}$ and $W = \{3,4,5,6,7\}$, and assume that $g = (1,2)$ and $h = (3,4,6)(5,7)$. We write $ab = (a,b)$ for the elements in $V \times W$ and get the cycle decomposition for (g,h) as

$$(13, 24, 16, 23, 14, 26)(15, 27)(25, 17)$$

and hence the cycle structure $6^1 2^2$. See Exercise 5 in Notes 2 for this course for more examples.

An important observation is that we can identify a coloring $c : \{1,\ldots,m\} \times \{1,\ldots,n\} \to S$ with the $m \times n$-matrix where the element in row i and column j is $c(i,j)$ for $1 \leq i \leq m$ och $1 \leq j \leq n$. In the Exercises Ö.8 and Ö.9 we study binary matrices under group action.

3 Burnside's lemma (Orbit-counting lemma)

Let G be a permutation group on a finite set V, and assume that G acts on a set X of colorings of V. Burnside's lemma gives a very convenient way of counting the number of orbits (equivalence classes) in X induced by the action of G. Let Y be an orbit in X. For every $c, c' \in Y$, let

$$G(c,c') = \{g : cg = c'\}.$$

Burnside's lemma is based on the following identity, that will be proved later.

$$|G(c, c')| = |G|/|Y|. \tag{1}$$

The identity implies that $|G(c, c)| = |G|/|Y|$ for all $c \in Y$, and hence

$$\sum_{c \in Y} |G(c, c)| = |G|.$$

Thus we arrive at

$$\sum_{c \in X} |G(c, c)| = |G| \cdot t, \tag{2}$$

where t is the number of orbits. This is because every orbit adds $|G|$ to the sum, and we have t orbits. The sum in (2) counts all pairs (g, c) such that $g \in G$, $c \in X$ and $cg = c$. The trick in Burnside's lemma is to sum over G instead of X. The set $G(c, c)$ is often called the *stabilizer* of c. Define also for $g \in G$, $\text{fix}(g)$ to be the number of colorings $c \in X$ such that $cg = c$.

Burnside's lemma *Then number of orbits in of a permutation group G acting on X is*

$$\frac{1}{|G|} \sum_{g \in G} \text{fix}(g).$$

Proof. In the sum $\sum_{g \in G} \text{fix}(g)$ we count every pair (g, c) such that $cg = c$ exactly once. This is the same thing as is counted in (2), hence the sum is equal to G times the number of orbits in X. Division by $|G|$ gives the desired result.

It only remains to prove formula (1). Let c and c' belong to the same orbit Y, and let $g \in G(c, c')$. Thus g satisfies $cg = c'$; such a g exists by the definition of orbit. Define $\varphi : G(c, c) \to G(c, c')$ as $\varphi(h) = hg$. This is a function from $G(c, c)$ to $G(c, c')$, since

$$c\varphi(h) = c(hg) = (ch)g = cg = c'$$

if $h \in G(c, c)$. Further we see that φ is a bijection, since φ has an inverse ψ defined by $\psi(h) = hg^{-1}$. Thus $|G(c, c)| = |G(c, c')|$. As a consequence we get for all $c, c' \in Y$ that

$$|G| = \sum_{c'' \in Y} |G(c, c'')| = \sum_{c'' \in Y} |G(c, c)| = |Y| \cdot |G(c, c)| = |Y| \cdot |G(c, c')|,$$

which gives (1). This ends the proof of Burnside's lemma. □

We have the following rather obvious condition on when $cg = c$:

Lemma 3.1 *For every $g \in G$ and $c \in X$ we have that $cg = c$ if and only if in the cycle decomposition of g every cycle is unicolored. That is, $c(v) = c(w)$ if v and w are two elements in V belonging to the same cycle in g.*

We now give some examples of usage of Burnside's lemma.

Example 1. Let $n \geq 1$, and consider two binary sequences $(b_0, \ldots, b_{n-1})$ and $(c_0, \ldots, c_{n-1})$ as equivalent if one can be obtained from the other by a cyclic shift. That is to say there is an r such that $b_i = c_{(i+r) \bmod n}$ for all i. The cyclic group C_n acts naturally on the set of binary sequences. If g denotes rotation with one step then g^r maps $(b_0, \ldots, b_{n-1})$ to $(b_r, b_{1+r}, \ldots, b_{n-1+r})$, where all indices are counted modulo n. Note that we may identify a binary sequence with a coloring of the vertices of a regular n-gon with colors 0 and 1. As described earlier C_n acts by rotation of the n-gon.

Let us use Burnside's lemma to count the number of equivalence classes of binary strings of length n with k 1:s. Let us study a group element g^r. It maps a binary string $(b_0, \ldots, b_{n-1})$ to itself if and only if $b_i = b_{(i+r) \bmod n}$ for all i. Set $d = \gcd(n, r)$, where we have defined $\gcd(n, 0) := n$. Using the same reasoning as in the proof of Lemma 15.4.1 in Cameron we see that g^r consists of d cycles each with n/d elements.

By Lemma 3.1 g^r fixes a coloring if and only if every cycle is unicoclored. If k is not divisible by n/d then there is thus no coloring with k 1:s that is kept fixed under the action of g^r. If k is divisible by n/d we get a coloring with k 1:s by choosing the color 1 to exactly $k/(n/d) = kd/n$ cycles. This can be done in $\binom{d}{kd/n}$ different ways. Write for simplicity $[n, r] = \gcd(n, r)$. Burnside's lemma gives the number of equivalence classes as

$$\frac{1}{n} \sum_r \binom{[n, r]}{k \cdot [n, r]/n},$$

where the sum is over all $r \in \{0, \ldots, n-1\}$ such that k is divisible by $n/[n, r]$.

As an explicit example, let $(n, k) = (6, 3)$.Then $k = 3$ is divisible by $n/[n, r] = 6/[6, r]$ if and only if r is even. The wanted number is thus

$$\begin{aligned} & \frac{1}{6}\left(\binom{[6,0]}{3 \cdot [6,0]/6} + \binom{[6,2]}{3 \cdot [6,2]/6} + \binom{[6,4]}{3 \cdot [6,4]/6}\right) \\ = \; & \frac{1}{6}\left(\binom{6}{3} + \binom{2}{1} + \binom{2}{1}\right) = \frac{20+2+2}{6} = 4. \end{aligned}$$

We may for instance take $(1,1,1,0,0,0), (1,1,0,1,0,0), (1,0,1,1,0,0), (1,0,1,0,1,0)$ as representatives for the four equivalence classes.

If instead $(n, k) = (6, 4)$ we get that $k = 4$ is divisible by $n/[n, r] = 6/[6, r]$ if and only if r is divisible by 3. Thus we get this time

$$\begin{aligned} & \frac{1}{6}\left(\binom{[6,0]}{4 \cdot [6,0]/6} + \binom{[6,3]}{4 \cdot [6,3]/6}\right) \\ = \; & \frac{1}{6}\left(\binom{6}{4} + \binom{3}{2}\right) = \frac{15+3}{6} = 3. \end{aligned}$$

Here we may chose $(1,1,1,1,0,0), (1,1,1,0,1,0), (1,1,0,1,1,0)$ as representatives for the equivalence classes.

See Exercise Ö.1 for more examples.

Example 2. This example is similar to the example in 15.2 in Cameron, but we study the colorings of the vertices of instead of the faces. We want to count the number of ways to color the vertices of a cube with r colors, where to colorings are considered identical if one can be transformed to the other by rotation of the cube. Let g be a group element with k cycles. To compute $\mathrm{fix}(g)$ we note that by Lemma 3.1 we have eactly r choices for each of the k cycles. Multiplication gives $\mathrm{fix}(g) = r^k$. By Section 2 we have one element of cycle structure 1^8, nine elements with structur 2^4, six elements with structure 4^2 and eight with structure $3^2 1^2$. We sum up the computation in the following table.

Cycle structure	Number g	$\mathrm{fix}(g)$
1^8	1	r^8
2^4	9	r^4
4^2	6	r^2
$3^2 1^2$	8	r^4

Burnside's lemma gives thus that the number of orbits with equal colorings is

$$\frac{1}{24}\left(r^8 + 9\cdot r^4 + 6\cdot r^2 + 8\cdot r^4\right) = \frac{r^2(r^6 + 17r^2 + 6)}{24}.$$

In the special case $r = 2$ we get the number of orbits as

$$\frac{2^2(2^6 + 17\cdot 2^2 + 6)}{24} = \frac{64 + 68 + 6}{6} = \frac{138}{6} = 23.$$

4 Cycle index

Let as above V be a finite set, X is a set of colorings and G a permutation group on V acting on X. Since G preserves the number of occurrences of a given color we can study the subset of X consisting of all colorings with a given number of occurrences of each color. In this section we will only study 2-colorings with 0 and 1. This is equivalent to study subsets of V, where a given coloring correspond to the subset colored with 1.

Let X_k be the subset of X consisting of the colorings having exactly k 1:s. For a given group element g, let $\mathrm{fix}_k(g)$ be the number of colorings in X_k that are kept fixed by g. Burnside's lemma gives that the number of orbits in X_k is

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{fix}_k(g).$$

Define

$$\mathrm{fix}(g;t) = \sum_{k=0}^{n}\mathrm{fix}_k(g)t^k.$$

Then we get

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{fix}(g;t) = \frac{1}{|G|}\sum_{g\in G}\sum_{k=0}^{n}\mathrm{fix}_k(g)t^k = \sum_{k=0}^{n} t^k \frac{1}{|G|}\sum_{g\in G}\mathrm{fix}_k(g).$$

This implies that the coefficient for t^k in this sum is the number of orbits of colorings with exactly k 1:s.

We will now consider the polynom $\mathrm{fix}(g;t)$ in the case where X is the set of possible 2-colorings, i.e. the set of functions from V to $\{0,1\}$. Suppose that g has cycle structure $1^{e_1}2^{e_2}\cdots n^{e_n}$. Introduce variables $s_1, s_2, \ldots, s_n$, and define g:s *cycle index* to be the polynomial

$$z(g; s_1, s_2, \ldots, s_n) = s_1^{e_1} s_2^{e_2} \cdots s_n^{e_n}.$$

A coloring is fixed by g if and only if every cycle is unicolored. This means that a cycle of length i has either 0 or i elements with the color 1. Multiplication principle gives

$$\mathrm{fix}(g;t) = z(g; 1+t, 1+t^2, \ldots, 1+t^n).$$

As you see we just replace s_i with $1+t^i$ in the formula for $z(g; s_1, s_2, \ldots, s_n)$. The coefficient for t^k is then exactly the number of ways of getting k elements by choosing none or all elements from each cycle in g.

Now define

$$Z(G; s_1, s_2, \ldots, s_n) = \frac{1}{|G|} \sum_{g \in G} z(g; s_1, s_2, \ldots, s_n).$$

This is the *cycle index* of the entire group G.

Theorem 4.1 (Cycle index theorem) *Let a_k be the number of orbits of colorings with exactly k 1:s. Then*

$$\sum_{k=0}^{n} a_k t^k = Z(G; 1+t, 1+t^2, \ldots, 1+t^n).$$

This follows from Burnside's lemma, since by the reasoning above

$$\begin{aligned} Z(G; 1+t, 1+t^2, \ldots, 1+t^n) &= \frac{1}{|G|} \sum_{g \in G} z(g; 1+t, 1+t^2, \ldots, 1+t^n) \\ &= \frac{1}{|G|} \sum_{g \in G} \mathrm{fix}(g;t). \end{aligned}$$

We now return to the examples of Section 3.

Example 1. Recall the first example of Section 3 with binary strings of length n. As we concluded there a group element g^r consists of d cycles each of length n/d, where $d = \gcd(n,r)$. Recall also that $\gcd(n,0) = n$. This implies that

$$z(g^r; s_1, \ldots, s_n) = s_{n/d}^d$$

and thus

$$Z(C_n; s_1, \ldots, s_n) = \frac{1}{n} \sum_{r=0}^{n-1} s_{n/\gcd(n,r)}^{\gcd(n,r)}.$$

Let a_k be the number of orbits of binary strings with exactly k 1:s. The Cycle index theorem gives us that

$$\sum_{k=0}^{n} a_k t^k = Z(C_n; 1+t, \ldots, 1+t^n) = \frac{1}{n}\sum_{r=0}^{n-1}(1+t^{n/\gcd(n,r)})^{\gcd(n,r)}.$$

Specifying to for example $n = 6$ we get

$$\begin{aligned}
\sum_{k=0}^{6} a_k t^k &= \frac{1}{6}\sum_{r=0}^{5}(1+t^{6/\gcd(6,r)})^{\gcd(6,r)} \\
&= \frac{1}{6}\left((1+t)^6 + (1+t^2)^3 + 2(1+t^3)^2 + 2(1+t^6)\right) \\
&= \frac{1}{6}\left(6 + 6t + 18t^2 + 24t^3 + 18t^4 + 6t^5 + 6t^6\right) \\
&= 1 + t + 3t^2 + 4t^3 + 3t^4 + t^5 + t^6.
\end{aligned}$$

From this we can easily read of the number of orbits with k 1:s for each k.

Example 2. Recall the example with the colorings of the corners of the cube in Section 3. We let $r = 2$ and restrict our selves to study 2-colorings. Let G denote the rotation group of the cube. By our calculations in Section 3 we get that

$$Z(G; s_1, s_2, \ldots, s_n) = \frac{1}{24}\left(s_1^8 + 9s_2^4 + 6s_4^2 + 8s_3^2 s_1^2\right).$$

This implies

$$\begin{aligned}
& Z(G; 1+t, 1+t^2, \ldots, 1+t^n) \\
= \;& \frac{1}{24}\left(1\cdot(1+t)^8 + 9\cdot(1+t^2)^4 + 6\cdot(1+t^4)^2 + 8\cdot(1+t^3)^2(1+t)^2\right) \\
= \;& 1 + t + 3t^2 + 3t^3 + 7t^4 + 3t^5 + 3t^6 + t^7 + t^8.
\end{aligned}$$

The Cycle index theorem says that the number of colorings with exactly k 1:s is equal to the coefficient of t^k, which now can be easily found in the expansion above. Note also that the sum of all coefficients is 23, which is the answer we got in Section 3.

Ö Övningsuppgifter

These supplementary exercises are unfortunately still in Swedish. Most of them should be reasonably easy to understand anyway. You are welcome to ask questions if there is something you don't understand.

Ö.1 Cykliska binära följder, del 1

Låt $n \geq 1$, och betrakta två binära följder av längd n som ekvivalenta om den ena kan erhållas från den andra via en rotation; se Exempel 1 i avsnitt 3. Beräkna antalet banor (ekvivalensklasser) av följder med exakt k ettor i följande fall:

(a) $(n, k) = (8, 4)$.

(b) $(n, k) = (9, 6)$.

(c) Godtyckligt par (n, k) sådant $1 \leq k < n$ och $\gcd(n, k) = 1$.

Svårighetsgrad: D

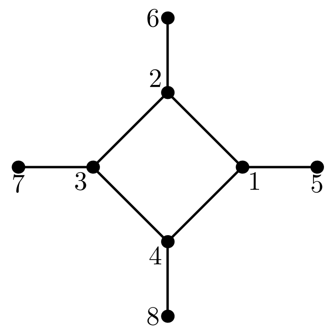
Ö.2 Par av hörn i en kvadrat med fyra löv

Den dihedrala gruppen D_4 verkar på grafen H i Figur Ö.2 genom rotation och spegling. Låt X vara mängden av 2-färgningar av H som uppfyller följande villkor:

- Exakt två hörn har färgen 1, och dessa två hörn är inte sammanbundna.

(Detta innebär att de två hörnen med färgen 1 utgör en oberoende mängd; se övningsuppgift 20 i häftet till del III.) Beräkna antalet banor av tillåtna färgningar.

Svårighetsgrad: C



Figur 2: Grafen H i uppgift Ö.2.

Ö.3 Kantfärgningar av kuben, del 1

I avsnitt 2 beskrev vi hur rotationer av kuben ger upphov till grupper av permutationer dels av kubens åtta hörn och dels av kubens sex sidor. I denna uppgift betraktar vi motsvarande grupp av permutationer av kubens tolv kanter.

(a) Ange cykelstrukturen för de 24 gruppelementen med avseende på hur de verkar på de tolv kanterna.

(b) Två färgningar av kanterna är ekvivalenta om den ena färgningen kan överföras i den andra via en rotation av kuben. För $r \geq 1$, använd Burnsides lemma för att beräkna antalet banor av färgningar av kanterna med färgerna $\{1, \ldots, r\}$. Vad blir antalet för $r = 2$? Vi har inga restriktioner angående färgningen av intilliggande kanter.

Svårighetsgrad: B

Ö.4 Samband mellan färgningar och urvalsproblem

Visa att vi har bijektioner mellan följande par:

(a) – Märkta r-färgningar av den kompletta grafen K_n.
- Följder av längd n utan upprepning med element från en mängd med r element.

(b) – Omärkta r-färgningar av den kompletta grafen K_n.
- Delmängder av storlek n till en mängd med r element.

(c) – Märkta r-färgningar av den tomma grafen O_n.
- Följder av längd n med element från en mängd med r element (upprepning är tillåten).

(d) – Omärkta r-färgningar av den tomma grafen O_n.
- Multimängder av storlek n över en mängd med r element.

(Se avsnitt 1 i häfte I för detaljer om de fyra olika typerna av följder.)

Svårighetsgrad: D

Ö.5 Omärkta färgningar av tomma grafer

Använd (d) i uppgift Ö.4 för att visa att

$$\frac{1}{n!}\sum_{\pi\in S_n} r^{\gamma(\pi)} = \binom{n+r-1}{n},$$

där $\gamma(\pi)$ är lika med antalet cykler i cykeluppdelningen för π. (Koefficienterna i polynomet $\sum_{\pi\in S_n} r^{\gamma(\pi)}$ är absolutbeloppen av Stirlingtalen av första slaget.)

Svårighetsgrad: D

Ö.6 Cykliska binära följder, del 2

Använd samma beteckningar som i uppgift Ö.1 och Exempel 1 i avsnitt 4. För $0 \le k \le n$, beräkna antalet banor av följder med exakt k ettor i följande fall:

(a) $n = 8$.

(b) $n = 9$.

(c) n godtyckligt primtal.

Svårighetsgrad: C

Ö.7 Kantfärgningar av kuben, del 2

Använd samma beteckningar som i uppgift Ö.3. För $0 \le k \le 12$, beräkna antalet ekvivalensklasser av 2-färgningar av de tolv kanterna i kuben sådana att exakt k kanter får färgen 1.

Svårighetsgrad: B

Ö.8 Matriser under gruppverkan, del 1

Betrakta två $m\times n$-matriser A och B som ekvivalenta om man kan erhålla B från A genom att kasta om ordningen på raderna och på kolonnerna. Observera att vi kan identifiera en binär $m \times n$-matris med en färgning av mängden $\{1,\ldots,m\} \times \{1,\ldots,n\}$; se avsnitt 2.

(a) Använd en direkt metod för att beräkna antalet ekvivalensklasser av binära 2×3-matriser med k ettor för $0 \leq k \leq 6$.

(b) Ange en gruppverkan vars banor svarar mot ekvivalensklasserna.

(c) Lös uppgift (a) med hjälp av cykelindexsatsen.

Svårighetsgrad: (a): E, (b): B, (c): B

Ö.9 Matriser under gruppverkan, del 2

Betrakta nu två $m \times n$-matriser A och B som ekvivalenta om man kan erhålla B från A genom att rotera raderna och kolonnerna cykliskt.

(a) Ange en gruppverkan vars banor svarar mot ekvivalensklasserna.

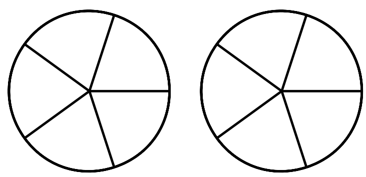
(b) Beräkna antalet banor av binära 3×4-matriser med k ettor för $0 \leq k \leq 12$.

Svårighetsgrad: B

Ö.10 Två cirkelskivor indelade i fem tårtbitar

Två cirkelskivor är indelade i fem identiska tårtbitar som i Figur 3. Vi kan rotera var och en av cirkelskivorna med heltalsmultiplar av 360/5 grader, och vi kan kombinera varje tänkbar rotation med att byta plats på de två skivorna. Detta ger en verkan på de två cirkelskivorna med en grupp G av storlek $2\cdot 2^5 = 50$. För $0 \leq k \leq 10$, beräkna antalet ekvivalensklasser av mängder med k tårtbitar, där två mängder A och B betraktas som ekvivalenta om det finns ett gruppelement som avbildar A på B.

Svårighetsgrad: A



Figur 3: Två cirkelskivor indelade i fem identiska tårtbitar.