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SF2715 Applied Combinatorics
May 25, 2011

Max is 24 points and then possibly 2 extra bonus points for the last problem and the bonus points you have gathered for the hand inproblems. Grades will be given as follows; 21 or more gives an A; 18-20 B; 16-17 C; 14-15 D; 11-13 E and 10 will give Fx.
Skrivtid: 8.00-13.00.
Hjälpmedel: Allowed equipment: pencil, eraser, ruler.
Please motivate your solutions clearly!

1. Consider the following sum:

$$e_1 + e_2 + 3e_3 + 7e_4 + 19e_5 + 29e_6,$$

where $e_i = 0$ or 1 for all $1 \le i \le 6$. Let F_n be the number of ways that this sum could be n. Write the generating function $\sum_{n=0}^{60} F_n x^n$ as a product. (2 points)

2. Determine which permutation gives the following pair of standard Young tableaux under the RSK-correspondence. (2 points)

$$\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline 3&7&10\\ \cline{1-3} 6&9\\ \cline{1-2} 8\\ \cline{1-1}\end{array} \quad , \quad \begin{array}{|c|c|c|c|}\hline 1&2&5&8\\ \hline 3&4&9\\ \cline{1-3} 6&7\\ \cline{1-2} 10\\ \cline{1-1}\end{array}$$

3. Draw the tree on 10 vertices with the Prüfer code $(6, 5, 5, 6, 2, 1, 6, 3)$. (2 points)

4. Simplify the following sum to a formula involving only one or two binomial coefficients.

$$\sum_{k \ge -2} \frac{1}{k+3} \binom{49}{k+2} \binom{100}{70-k}$$

(3 points)

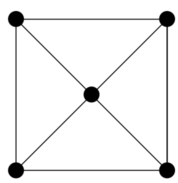
5. Consider the recursion

$$a_{n+2} = 5a_{n+1} - 6a_n + 1, \quad n \ge 0$$

with initial values $a_0 = 1, a_1 = 3$.

   (a) Determine the ordinary generating function for a_n.
   (b) Use it to determine an exact formula for a_n. (3 points)

6. The *edges* of the following graph shall be colored with two colors blue and red. For each k determine how many non-equivalent colorings there are with k red edges. Two colorings are considered equivalent if we can obtain one from the other by rotation or reflection. There are no other restrictions on the coloring. (4 points)



7. For $n \geq 2$, let $d_1, \ldots, d_n$ be a sequence of positive integers such that $\sum_i d_i = 2n - 2$. Prove that the number of trees on the labeled nodes $1, \ldots, n$ where node i has valency d_i for each i is equal to

$$\frac{(n-2)!}{(d_1 - 1)! \ldots (d_n - 1)!}.$$

Hint: A tree always has a leaf, what happens if you remove it? (4 points)

8. Assume that C is a 3-error correcting binary code of length 23, (that is $q = 2, n = 23, e = 3, d = 7$ with the notations in the book). Assume also that the zero word $\mathbf{0}$ is in C and that $|C| = 2^{12}$. Let A_i be the number of codewords in C that has weight i, so in particular $A_0 = 1$.

   (a) Prove that C is a perfect code. (2 points)

   (b) Prove that $A_7 = 253$. Hint: Count in two different ways the number of pairs $(\mathbf{x}, \mathbf{c})$ where $\mathbf{x}$ is any word with weight 4 and $\mathbf{c} \in \mathbf{C}$ at distance 3 from $\mathbf{x}$. (2 points)

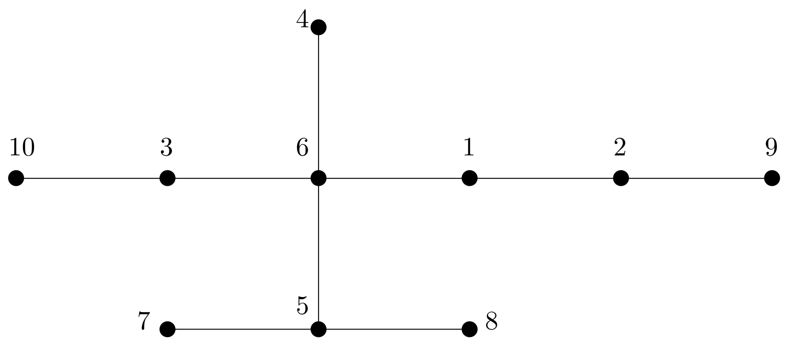
   (c) Bonus question: Deduce also A_8 and explain why all A_i are uniquely determined. (2 points)

Solutions

1. If e_6 is 0 we get no contribution to the sum, if $e_6 = 1$ we get a contribution of 27. Thus it corresponds to a factor $1+x^{27}$. Likewise with the other e_i so we get $\sum_{n=0}^{60} F_n x^n = (1+x)^2(1+x^3)(1+x^7)(1+x^{19})(1+x^{27})$.

2. Using the description of the inverse of RSK found in the text book we get the permutation 8 9 1 6 7 3 4 10 5 2. (In the students' solutions I expect to see some description of some intermediate steps.)

3. Following the description of the inverse of Prüfer coding we get the following tree. (In the students' solutions I expect to see some explanation of how you have proceeded.)



4. First use "absorption/extraction" identity $\frac{1}{k+3}\binom{49}{k+2} = \frac{1}{50}\binom{50}{k+3}$. Then use Vandermond's convolution $\sum_{k \geq -3} \binom{50}{k+3}\binom{100}{70-k} = \binom{150}{73}$. Putting this together we get

$$\sum_{k \geq -2} \frac{1}{k+3}\binom{49}{k+2}\binom{100}{70-k} = \frac{1}{50}\left(\binom{150}{73} - \binom{100}{73}\right).$$

5. (a) Let $f(x) = \sum_{n \geq 0} a_n x^n$. Multiplying each side of the recursion with x^n and summing over $n \geq 0$ we get

$$\frac{f(x) - a_1 x - a_0}{x^2} = \frac{5(f(x) - a_0)}{x} - 6f(x) + \frac{1}{1-x}.$$

   This simplifies to

$$f(x) = \frac{1}{1-3x} + \frac{x^2}{(1-x)(1-2x)(1-3x)}.$$

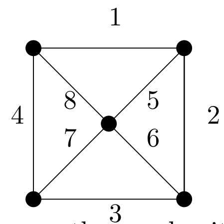
(b) By partial fraction expansion (for instance “handpåläggning”) this gives

$$f(x) = \frac{1/2}{1-x} - \frac{1}{1-2x} + \frac{3/2}{1-3x}.$$

From this we can deduce the exact formula

$$a_n = \frac{1}{2}(1 + 3^{n+1}) - 2^n.$$

6. Give the following labels to the edges



We have the dihedral group D_4 acting on the graph with the following possible permutations

permutation	Cycle index
identity	s_1^8
(1 2 3 4)(5 6 7 8)	s_4^2
(1 4 3 2)(5 8 7 6)	s_4^2
(1 3) (2 4)(5 7)(6 8)	s_2^4
(1)(3)(2 4)(5 8)(6 7)	$s_1^2 s_2^3$
(2)(4)(1 3)(5 6)(7 8)	$s_1^2 s_2^3$
(5)(7)(1 2)(3 4)(6 8)	$s_1^2 s_2^3$
(6)(8)(1 4)(2 3)(5 7)	$s_1^2 s_2^3$

Thus the cycle index for the entire group is $Z_G = \frac{1}{8}(s_1^8 + 2s_4^2 + s_2^4 + 4s_1^2 s_2^3)$. Let a_k be the number of ways to color the graph with k red vertices. By the Cycle index theorem we get by substituting $1 + t^i$ for s_i in Z_G

$$\begin{aligned}\sum_{k=0}^{8} a_k t^k &= \frac{1}{8}\left((1+t)^8 + 2(1+t^4)^2 + (1+t^2)^4 + 4(1+t)^2(1+t^2)^3\right) \\ &= 1 + 2t + 6t^2 + 10t^3 + 13t^4 + 10t^5 + 6t^6 + 2t^7 + t^8.\end{aligned}$$

7. We prove this by induction over n. For $n = 2$ the only possible degrees are $d_1 = d_2 = 1$ and the formula becomes $\frac{0!}{0!0!} = 1$ (remember that $0! = 1$), which is true since there are exactly one such tree.

Assume now we are given a degree sequence $d_1, \dots, d_n$, such that $d_i \geq 1$ and $\sum_i d_i = 2n - 2$. All the d_i cannot be greater than 2 (by the pigeon hole principle) so there has to be one (in fact at least two) that is equal to 1. We can without loss of generality assume that $d_n = 1$. This means that n is a leaf in every tree with degree sequence $d_1, \dots, d_n$. Let j denote the node that n is adjacent to. Removing the leaf n cause the degree to go down by one for node j and give degree sequence $d_1, \dots, d_{j-1}, d_j - 1, d_{j+1}, \dots, d_{n-1}$. By induction, we know that, if $d_j - 1 \geq 1$ the number of trees on $n - 1$ nodes with that degree sequence is

$$\begin{aligned}&\frac{(n-3)!}{(d_1-1)!\dots(d_{j-1}-1)!(d_j-2)!(d_{j+1}-1)!\dots(d_{n-1}-1)!} \\ =&\frac{(d_j-1)(n-3)!}{(d_1-1)!\dots(d_{j-1}-1)!(d_j-1)!(d_{j+1}-1)!\dots(d_{n-1}-1)!}\end{aligned}$$

Note that the second formula is valid also if $d_j = 1$. Summing over all j gives that the total number of trees is

$$\begin{aligned}&\sum_j \frac{(d_j-1)(n-3)!}{(d_1-1)!\dots(d_{n-1}-1)!} = \frac{(n-3)!}{(d_1-1)!\dots(d_{n-1}-1)!}\sum_j (d_j - 1) \\ =&\frac{(n-3)!}{(d_1-1)!\dots(d_{n-1}-1)!}(n-2) = \frac{(n-2)!}{(d_1-1)!\dots(d_{n-1}-1)!}.\end{aligned}$$

Since $d_n = 1$ this is the equal to the formula we want to prove.

8. (a) The Hamming bound (sphere packing bound) says that

$$|C| \leq \frac{2^{23}}{\sum_{i=0}^{3} \binom{23}{i}} = \frac{2^{23}}{1 + 23 + 253 + 1771} = \frac{2^{23}}{2048} = 2^{12}.$$

Equality in the Hamming bound is the definition of a perfect code and thus C is perfect.

(b) We follow the given hint. First note that $\mathbf{0} \in C$ and $d = 7$ implies that there are no code words in C of weights 1 to 6. The total number of $\mathbf{x} \in H(23, 2)$ with $wt(\mathbf{x}) = \mathbf{4}$ is $\binom{23}{4}$. Since C is assumed to have minimum distance 7 each such $\mathbf{x}$ can be at distance 3 from at most one code word $\mathbf{c}$. Since C is assumed to be perfect, there is in fact exactly one such $\mathbf{c} \in C$ and thus exactly one pair $(\mathbf{x}, \mathbf{c})$ for any given $\mathbf{x}$. One the other hand we can choose A_7 codewords $\mathbf{c}$ with $wt(\mathbf{c}) = 7$ and then we can find $\binom{7}{4}$ words $\mathbf{x}$ of weight 4 using using exactly 4 of the 7 ones in $\mathbf{c}$. This double counting gives $A_7\binom{7}{4} = \binom{23}{4}$. Since $\binom{23}{4} = \frac{23\cdot 22\cdot 21\cdot 20}{4\cdot 3\cdot 2} = 23 \cdot 11 \cdot 7 \cdot 5$ and likewise $\binom{7}{4} = 7 \cdot 5$ we conclude

$$A_7 = \frac{23 \cdot 11 \cdot 7 \cdot 5}{7 \cdot 5} = 253.$$

(c) To determine A_8 we count all words of weight 5 by which code word they are closest to, which could be a word of weight 7 or 8. For each code word $\mathbf{c}$ of weight 7 there are $\binom{7}{5}$ words of weight 5 at distance 2 from $\mathbf{c}$ and likewise for a code word of weight 8 there are $\binom{8}{5}$ words of weight 5 at distance 3. Since C is perfect and 3-error correcting this gives the relation $\binom{23}{5} = A_7\binom{7}{5} + A_8\binom{8}{5}$. From this we can deduce $A_8 = 506$.

To determine A_k we proceed similarly by counting the words of weight $k - 3$. In general we get

$$\begin{aligned}\binom{23}{k-3} &= A_{k-6}\binom{23-k+6}{3} + A_{k-5}\binom{23-k+5}{2} + A_{k-4}\left(\binom{23-k+4}{1} + \binom{23-k+4}{2}(k-4)\right) \\ &\quad + A_{k-3}\left(1 + (k-3)(23-k+3)\right) + A_{k-2}\left(\binom{k-2}{1} + \binom{k-2}{2}(23-k+2)\right) \\ &\quad + A_{k-1}\binom{k-1}{2} + A_k\binom{k}{3}.\end{aligned}$$

Let us look at the term containing A_{k-4}. For each code word of weight $k-4$ we may obtain a word of weight $k-3$ by either changing one 0 to a 1 or by changing two 0 to 1 and one 1 to 0. Similarly for the other terms. This identity enables us to recursively compute A_k for all $0 \leq k \leq 23$.