

Final Exam
SF2715 Applied Combinatorics, 6 hp
 Wednesday, 17 August 2011, 14:00–19:00

Solutions

1. If $x_2 = k$, then it will contribute $3k$ to the value of n . Since x_2 must be positive, it contributes a factor $(x^3 + x^6 + x^9 + x^{12} + \dots)$ to the generating function. The other integers x_1, x_3, x_4 contribute similarly, so the generating function $\sum_n a_n x^n$ is equal to

$$\begin{aligned} & (x + x^2 + x^3 + \dots)(x^3 + x^6 + x^9 + \dots)^2(x^7 + x^{3 \cdot 7} + x^{5 \cdot 7} + \dots) \\ &= \frac{x}{1-x} \left(\frac{x^3}{1-x^3} \right)^2 \frac{x^7}{(1-x^{14})} = \frac{x^{14}}{(1-x)(1-x^3)^2(1-x^{14})}. \end{aligned}$$

2. By the hook length formula (page 217 in Cameron) the answer is

$$\frac{8!}{6 \cdot 4 \cdot 1 \cdot 4 \cdot 2 \cdot 3 \cdot 1 \cdot 1} = \frac{8 \cdot 7 \cdot 5}{4} = 70.$$

3. The Matrix-tree theorem states that the number of spanning trees is equal to the value of any cofactor of the matrix $D - A$, where A is the adjacency matrix and D the matrix with the degrees on the diagonal. In this case we have

$$D - A = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

We delete the last row and column of $D - A$ and take the determinant

$$\begin{aligned} \begin{vmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} &= 1 \cdot \begin{vmatrix} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 2 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} \\ &= (1 - 6) + 2 \cdot (18 - 3 - 2) = 21. \end{aligned}$$

Answer: There are 21 spanning trees.

(Partial credit if you solved the problem with other methods.)

4. Assume we are given a set partition of the described type.

If n belongs to a block with exactly two elements, then there are $n - 1$ possibilities for the other element in that block. There are $b(n - 2, k - 1)$ ways to partition the remaining $n - 2$ elements into $k - 1$ blocks, each of size at least two.

If n is in a block of size greater than two, then we may just remove n and obtain a set partition of $n - 1$ elements in k blocks, each of size at least two. The number n could have been in any of the k blocks.

Since these two cases are disjoint and include all possibilities, we get the recursion

$$b(n, k) = (n - 1) \cdot b(n - 2, k - 1) + k \cdot b(n - 1, k),$$

which is valid for all $n \geq 3, k \geq 2$.

It is evident that $b(n, 1) = 1$ for all $n \geq 2$. Since each block contains at least two elements, we also have that $b(n, k) = 0$ for all $n \leq 2k - 1$. These two sets of boundary values are easily seen to determine $b(n, k)$ for any value of $n, k \geq 1$ using the recursion given.

5. The exponential generating function $\sum_n w_n x^n / n!$ equals

$$\begin{aligned} & \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) \left(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \\ &= (e^x - 1 - x) \cdot \frac{e^x + e^{-x}}{2} \cdot \frac{e^x - e^{-x}}{2} = (e^x - 1 - x) \cdot \frac{e^{2x} - e^{-2x}}{4} \\ &= \frac{1}{4} (e^{3x} - e^{-x} - e^{2x} + e^{-2x} - xe^{2x} + xe^{-2x}). \end{aligned}$$

Studying the coefficients of each of the functions involved, we conclude that

$$w_n = \frac{1}{4} (3^n - (-1)^n - 2^n + (-2)^n - n2^{n-1} + n(-2)^{n-1}).$$

Equivalently,

$$w_n = \begin{cases} \frac{1}{4} (3^n - 1 - n \cdot 2^n), & \text{if } n \text{ is even} \\ \frac{1}{4} (3^n + 1 - 2^{n+1}), & \text{if } n \text{ is odd} \end{cases}$$

6. A direct reformulation is that d_n counts the partitions of n whose Ferrer diagram has no rows of the same length. Second, note that if columns k and $k + 1$ have the same length, then there is no part of size k . Thus, e_n counts the partitions of n whose Ferrer diagram has no two columns of the same length. Hence conjugation gives a bijection between the integer partitions counted by d_n and e_n .

7. The dihedral group D_4 acts on the toy as described by the following table.

permutation	Cycle index
identity	$(s_1)^8$
Rotation $\pm 90^\circ$ (two cases)	$(s_4)^2$
Rotation 180°	$(s_2)^4$
Reflection vertically	$(s_2)^4$
Reflection horizontally	$(s_2)^4$
Reflection diagonally (two cases)	$(s_2)^4$

Thus the cycle index for the entire group is $Z_G = \frac{1}{8}((s_1)^8 + 2(s_4)^2 + 5(s_2)^4)$. Let a_k be the number of ways to assign the color red to k of the triangles. Substituting $1 + t^i$ for s_i in Z_G , the Cycle Index Theorem yields that

$$\begin{aligned} \sum_{k=0}^8 a_k t^k &= \frac{1}{8} ((1+t)^8 + 2(1+t^4)^2 + 5(1+t^2)^4) \\ &= 1 + t + 6t^2 + 7t^3 + 13t^4 + 7t^5 + 6t^6 + t^7 + t^8. \end{aligned}$$

8. (a) By the max-flow min-cut theorem, a flow of size $\sum a_{i,j}$ exists if and only if this is the smallest size of a cut. Since the edges in the middle have infinity capacity, each edge in such a minimal cut must be incident to either s or t . In a given cut, let I be the set of indices i such that the edge between s and p_i is contained in the cut. Then we must have that the edge between $q_{i,j}$ and t belongs to the cut whenever $i \notin I$ or $j \notin I$, because otherwise there is a path from s to t via $q_{i,j}$. Since we are looking for a cut of minimum size, we may assume that there are no other edges in the cut. Hence the size of the cut is

$$\sum_{i \in I} w_i + \sum_{i \notin I \text{ or } j \notin I} a_{i,j}.$$

We obtain the inequality

$$\sum_{i \in I} w_i + \sum_{i \notin I \text{ or } j \notin I} a_{i,j} \geq \sum_{i < j} a_{i,j} \iff \sum_{i \in I} w_i \geq \sum_{\{i,j\} \subseteq I} a_{i,j}.$$

To summarize, a flow of the given size exists if and only if

$$\sum_{i \in I} w_i \geq \sum_{\{i,j\} \subseteq I} a_{i,j} \tag{1}$$

for each $I \subseteq \{1, \dots, n\}$.

- (b) Consider the inequality (1) for a given set I . Let M_I be the total number of points for the players in the set $\{P_i : i \in I\}$. The right-hand side equals the number of games in which both players belong to the set. For each such game, the total number of points for the players in the set increases by one. In particular, we have that the right-hand side is less than or equal to M_I . For a flow with the desired value to exist, the left-hand side must be greater than or equal to M_I and hence greater than or equal to the right-hand side.