Proportionality and the mathematics of elections

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Introduction

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- Given votes for different parties in an election, how should one proportionally determine the number of seats each party should get in the parliament.

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A very similar question, well studied in the USA, is:

- Given the size of the population in the various states. How many seats should they each have in the house of representatives?

This later question has become known as the problem of **apportionment**.

Notation

Fix notation
$M :=$ total number of seats
$P :=$ total size of population
$p_i :=$ population in state i
m_i will be the number of seats given to state i

Hamilton-Hare

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- First each state gets as many seats as the integer part of τ_i.
- Then the remaining $M - \sum_i \lfloor \tau_i \rfloor$ seats are given to the states with largest decimal part of τ_i.

Example

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$\frac{p_i}{P}$	14%	43%	43%

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This is known as the **Alabama-paradox**, because it threatened Alabama in 1880.

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	Year 1900			Year 1901		
State	p_i	τ_i	m_i	p_i	τ_i	m_i
Virginia	$1,854,184$	9.599	10			
Maine	$694,4130$	3.595	3			
Total	$74,562,608$		386			

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The total grew with 2%.

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Reason: ”The $1,000,000$ people are a little more than the 5 seats.”

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Comparison	66.6	47.4($\frac{237}{5}$)	43.3	Seat 6 to A
Comparison	47.6($\frac{333}{7}$)	47.4	43.3	Seat 7 to A
Seats	4	2	1	Final apportionment

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State A is just above 3.5 seats and "should" be rounded up.
State B is just below 2.5 seats and "should" be rounded down.

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$$333 > \frac{333+237}{6} \cdot 3.5 \text{ and } 237 < \frac{333+237}{6} \cdot 2.5.$$

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which is exactly the comparison made by Webster's method.

A great advantage is of course that all these pairwise comparisons can be made simultaneously.

Divisor methods

Historically there are five divisor methods.

Name	Divisors	Formula
Webster	$\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$ $\frac{9}{2} \dots$	$m + \frac{1}{2}$

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Huntington-Hill	"0"	$\sqrt{2}$	$\sqrt{6}$	$\sqrt{12}$	$\sqrt{20}$ . . .	$\sqrt{m(m+1)}$
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They are ordered so Adams is best for small states and Jefferson best for large states.

Some US history

1792 – 1830	Jefferson
1840	Webster
1850 – 1870	Hamilton
1880 – 1910	Hamilton and Webster
1930 –	Huntington-Hill

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1948 another NAS group (Eisenhart, Morse, von Neumann) also suggested H-H.

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Their conclusion was: Webster is the most fair method!

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Theorem (Balinski - Young)

No divisor method can guarantee the quota requirement.

However, it is in practice violated very rarely by Webster's method.