SF2724: TOPICS IN MATHEMATICS IV: APPLIED TOPOLOGY HOMEWORK SET 2

MATTIAS DAHL

Let ω_{n-1} be the $(n-1)$-form on $\mathbb{R}^n \setminus \{0\}$ defined by

$$\omega_{n-1} = \frac{1}{r^n} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $\widehat{dx_i}$ is omitted and $r = (x_1^2 + \cdots + x_n^2)^{1/2}$.

(1) Show that ω_{n-1} is closed. Compute $dr \wedge \omega_{n-1}$.

(2) Let $F : \mathbb{R}^2 \to \mathbb{R}^3 \setminus \{0\}$ be given in spherical coordinates by

$$(y_1, y_2) \mapsto (\rho \sin\theta \cos\phi, \rho \sin\theta \sin\phi, \rho \cos\theta),$$

where ρ, θ, ϕ are smooth functions on $\mathbb{R}^2$, $\rho > 0$. Compute $F^*\omega_2$ expressed in $d\rho, d\theta, d\phi$.

(3) Let $U_1 = \mathbb{R}^n \setminus \{(x_1, 0, \ldots, 0) \mid x_1 \geq 0\}$ and $U_2 = \mathbb{R}^n \setminus \{(x_1, 0, \ldots, 0) \mid x_1 \leq 0\}$ so that $U_1 \cup U_2 = \mathbb{R}^n \setminus \{0\}$ and $U_1 \cap U_2 = \mathbb{R} \times (\mathbb{R}^{n-1} \setminus \{0\})$. Compute $\partial^*[\omega_{n-2}]$, where ∂^* is the connecting homomorphism in the Mayer-Vietoris sequence for $\{U_1, U_2\}$, $\partial^* : H^{n-2}(U_1 \cap U_2) \to H^{n-1}(\mathbb{R}^n \setminus \{0\})$.

Second updated hint: ω_{n-2} is considered as a form on $U_1 \cap U_2 = \mathbb{R} \times (\mathbb{R}^{n-1} \setminus \{0\})$ which is constant in the first factor. Explicitly, set $s = (x_2^2 + \cdots + x_n^2)^{1/2}$ so that $r^2 = x_1^2 + s^2$, and

$$\omega_{n-2} = \frac{1}{s^{n-1}} \sum_{j=2}^{n} (-1)^{j-1} x_j \, dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Choose a partition of unity $\{p_1, p_2\}$ subordinate to $\{U_1, U_2\}$ with functions of the form $\alpha(x_1/s)$. Since p_1 has support in U_1 (and thus vanishes on $\{(x_1, 0, \ldots, 0) \mid x_1 \geq 0\}$) the form $p_1\omega_{n-2}$ is well-defined on U_2, and vice versa $p_2\omega_{n-2}$ is well-defined on U_1. Define functions α_n by

$$\alpha_n(t) = c_n \int_{-\infty}^{t} \frac{1}{(1 + y^2)^{n/2}} \, dy,$$

do they give a proper partition of unity?

Some computations: The computation of $\partial^*[\omega_{n-2}]$ involves computing $dp_1 \wedge \omega_{n-2}$. With $p_1 = \alpha(x_1/s)$ we have

$$dp_1 = \alpha'(x_1/s) \frac{1}{s^3} \left(s^2 dx_1 - x_1 \sum_{i=2}^{n} x_i dx_i \right).$$

This gives

$$
\begin{aligned}
dp_1 \wedge \omega_{n-2} &= \alpha'(x_1/s)\frac{1}{s^3}\left(s^2 dx_1 - x_1 \sum_{i=2}^n x_i dx_i\right) \\
&\quad \wedge \frac{1}{s^{n-1}} \sum_{j=2}^n (-1)^{j-1} x_j \, dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\
&= \alpha'(x_1/s)\frac{1}{s^{n+2}}\left(s^2 \sum_{j=2}^n (-1)^{j-1} x_j \, dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \right.\\
&\quad \left. - x_1 \sum_{i,j=2}^n (-1)^{j-1} x_i x_j \, dx_i \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \right) \\
&= \alpha'(x_1/s)\frac{1}{s^{n+2}} \\
&\quad \left(s^2 \sum_{j=2}^n (-1)^{j-1} x_j \, dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n + s^2 x_1 dx_2 \wedge \cdots \wedge dx_n \right) \\
&= \alpha'(x_1/s)\frac{1}{s^n} \sum_{j=1}^n (-1)^{j-1} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\
&= \alpha'(x_1/s)\frac{r^n}{s^n}\omega_{n-1} \\
&= \alpha'(x_1/s)\frac{(x_1^2+s^2)^{n/2}}{s^n}\omega_{n-1} \\
&= \alpha'(x_1/s)\left(1 + \left(\frac{x_1}{s}\right)^2\right)^{n/2} \omega_{n-1}.
\end{aligned}
$$

With the choice of α above we have $\alpha'(t) = c_n(1+t^2)^{-n/2}$ and we conclude that

$$dp_1 \wedge \omega_{n-2} = c_n \omega_{n-1}.$$

Note that the constant c_n is determined by the fact that $\{p_1, p_2\}$ is a partition of unity, so in particular $\lim_{t\to\infty} \alpha_n(t) = 1$.