SF2724: TOPICS IN MATHEMATICS IV: APPLIED TOPOLOGY HOMEWORK SET 3

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(1) Let $U = \mathbb{R} \times (-1, 1)$ and define an action of $\mathbb{Z}$ on U by
$$\mathbb{Z} \times U \ni (n, (x, y)) \mapsto (x + n, (-1)^n y) \in U.$$
The quotient $M = U/\mathbb{Z}$ is the Möbius strip. Show that M does not have a volume form, and thus is not orientable.

(2) A symplectic form on a manifold M of dimension $2n$ is a closed 2-form ω such that
$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n} \neq 0$$
everywhere. Show that $H^{2k}(M) \neq 0$ for $k = 0, \ldots, n$ if M is a compact manifold which has a symplectic form. Show that S^2 is the only sphere which has a symplectic form.

(3) Let $\mathbb{H}$ be the quaternions and $S^3 \subset \mathbb{H}$ the unit sphere (for definitions see `http://en.wikipedia.org/wiki/Quaternion`). Compute the degree of the map $S^3 \ni x \mapsto x^n \in S^3$, for $n \in \mathbb{Z}$. Prove the fundamental theorem of algebra for quaternions: Every polynomial equation of the form
$$f(x) = a_0 x a_1 x a_2 \ldots a_{n-1} x a_n + \{\text{terms of lower order in } x\} = 0$$
where $n \geq 1$ and $a_i \neq 0$ has a solution. (Note: with more than one leading term this is no longer true, for example the equation $ix - xj = 1$ has no solution.)

Hint: You may use the fact that the equation $x^n = i$ has the same n solutions in $\mathbb{H}$ as it has in $\mathbb{C}$. Compute the degree of $x \mapsto x^n$ by computing the local index at the inverse images of the regular value i.

(4) Assume that f and g are Morse functions on the closed manifolds M and N respectively. Show that a Morse function h on $M \times N$ can be defined by $h(p, q) = f(p) + g(q)$. Describe the critical points and indices for h in terms of similar data for f and g. Derive the product formula for the Euler characteristics, $\chi(M \times N) = \chi(M)\chi(N)$. (This is Exercise 12.11 from the book.)

(5) Assume that M is a compact manifold with a Morse function f having only two critical points. What can be deduced about M from Lemmas 12.12 and 12.13 in the book?