Lecture 1

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Chapter 1

Complex vector bundles

The aim of this chapter is to introduce the main objects we are going to study: complex vector bundles. These are complex vector spaces parametrized by topological spaces. We will therefore discuss first topological spaces and continuous maps between them. The notion of compactness and a homotopy relation are going to be also presented. The definition of a vector bundle is the subject of the last part of this chapter.

1.1 Topological spaces and continuous maps

1.1.1 Definition. *A topological space is a set X together with a collection $\mathcal{T}$ of subsets of X which satisfies the following properties:*

(1) X and $\emptyset$ belong to $\mathcal{T}$.

(2) If U and V belong to $\mathcal{T}$, then so does the intersection $U \cap V$.

(3) If, for $s \in S$, U_s belong to $\mathcal{T}$, then so does the union $\bigcup_{s\in S} U_s$.

Let $(X, \mathcal{T})$ be a topological space. The collection $\mathcal{T}$ is called the **topology** of X and the members of $\mathcal{T}$ are called **open** subsets of X. A topological space $(X, \mathcal{T})$ will be often denoted simply by X, in which case the collection of open subsets $\mathcal{T}$ is assumed to be known. A subset $D \subset X$ is called **closed** if the complement $X \setminus D$ is open.

1.1.2 Excercise. Let $(X, \mathcal{T})$ be a topological space. Show that:

(1) X and $\emptyset$ are closed subset of X.

(2) If D and E are closed subset of X, then so is $D \cup E$.

(3) If, for $s \in S$, D_s is closed in X, then so is $\cap_{s\in S} D_s$.

1.1.3 Example. Let X be the set consisting of just one point. Such a set has a unique topology consisting of all the subsets of X. This topological space is denoted by Δ^0 or D^0 or $\mathbf{R}^0$ and is called the point.

1.1.4 Definition. *Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is called continuous if, for any open subset $V \subset Y$, the pre-image $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X.*

We will often use the term **map** for a continuous function between topological spaces. Thus any map is a function, but not all functions are maps, only those that are continuous. This notion of continuity is essential for understanding topological spaces. Maps will be used to compare spaces.

1.1.5 Excercise. Show that a function $f : X \longrightarrow Y$ is continuous if and only if, for any closed subset $D \subset Y$, the pre-image $f^{-1}(D) = \{x \in X \mid f(x) \in D\}$ is closed in X.

1.1.6 Excercise. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous functions between topological spaces. Show that the composition $gf : X \longrightarrow Z$ is also continuous.

1.1.7 Definition. *A continuous function $f : X \longrightarrow Y$ is called an isomorphism if there is a continuous function $g : Y \to X$ such that $fg = id_Y$ and $gf = id_X$. Two spaces X and Y are said to be isomorphic if there is an isomorphism $f : X \longrightarrow Y$.*

Note that an isomorphism of topological spaces is a one to one and onto function (such functions are also called bijections), as it has an inverse function. However even if a continuous function is a bijection (one to one and onto), so it has an inverse, the inverse may fail to be continuous. Thus to be an isomorphism it is not enough to be a continuous bijection. To be an isomorphism, in addition to having the inverse, this inverse has to be continuous.

1.1.8 Excercise. Let S be a set containing at least two distinct elements. Consider the following two collections of subsets of S: $\mathcal{T} = \{\emptyset, S\}$ and $\mathcal{D} = \{\text{all subsets of } S\}$. Show that $(S, \mathcal{T})$ and $(S, \mathcal{D})$ are topological spaces. Prove that $\mathrm{id} : S \longrightarrow S$ is a continuous function between $(S, \mathcal{D})$ and $(S, \mathcal{T})$, but it is not a continuous function between $(S, \mathcal{T})$ and $(S, \mathcal{D})$. Conclude that $\mathrm{id} : S \longrightarrow S$ is not an isomorphism between $(S, \mathcal{D})$ and $(S, \mathcal{T})$. Are $(S, \mathcal{D})$ and $(S, \mathcal{T})$ isomorphic?

Constructing new spaces

1.2 Subspaces

Let X be a topological space and $Y \subset X$ be a subset. Define $\mathcal{U}$ to be the collection of subsets of Y which consist of intersections $Y \cap U$ where U is an open subset in X. We call the collection $\mathcal{U}$ the subspace topology on Y.

1.2.1 Excercise. Let X be a topological space and $Y \subset X$ be a subset. Show that Y with the subspace topology is a topological space.

1.2.2 Excercise. Let X be a topological space and $Z \subset Y \subset X$ be subsets. Consider Y as a topological subspace of X. The set Z can be then considered as a subspace of Y and a subspace of X. Show that these two subspace topologies on Z are the same.

We will often use the above exercises to construct new topological spaces. We will define first an "ambient" topological space and then consider its subspaces as the new topological spaces.

1.2.3 Excercise. Let Y and X be topological spaces and $Z \subset X$ be a topological subspace. Show that the inclusion $i : Z \subset X$ is continuous. Show also that a function $f : Y \longrightarrow Z$ is continuous if and only if the composition $if : Y \longrightarrow X$ is continuous.

1.3 Disjoint unions

Let X and Y be topological spaces. Consider the disjoint union $X \coprod Y$ and the collection $\mathcal{U}$ of subsets $U \subset X \coprod Y$ such that $U \cap X$ is open in X and $U \cap Y$ is open in Y. We call the collection $\mathcal{U}$ the disjoint union topology on $X \coprod Y$.

More generally, for a collection of topological spaces $\{X_i\}_{i \in I}$, consider the disjoint union $\coprod_{i \in I} X_i$ and the collection $\mathcal{U}$ of subsets $U \subset \coprod_{i \in I} X_i$ such that $U \cap X_i$ is open for any $i \in I$. We call the collection $\mathcal{U}$ the disjoint union topology on $\coprod_{i \in I} X_i$.

1.3.1 Excercise. Let Y and X be topological spaces. Show $X \coprod Y$, with the disjoint union topology, is a topological space.

1.3.2 Excercise. Let X,Y, and Z be topological spaces. Show that a function $f : X \coprod Y \longrightarrow Z$ is continuous if and only if the compositions of f and the inclusions $\mathrm{in}_1 : X \subset X \coprod Y$ and $\mathrm{in}_2 : Y \subset X \coprod Y$ are both continuous.

1.3.3 Excercise. A space of the form $\coprod_I \Delta^0$ is called discreet. Show that any subset of a discreet space is open. Show that any function out of a discreet space is continuous. Show that any function into Δ^0 is continuous.

1.4 Products

Let X and Y be topological spaces. Consider the product $X \times Y$ and the collection $\mathcal{U}$ of subsets $U \subset X \times Y$ such that, for any point $(x, y) \in U$, there are open subsets $x \in U_1 \subset X$ and $y \in U_2 \subset Y$ such that $U_1 \times U_2 \subset U$. We call the collection $\mathcal{U}$ the product topology on $X \times Y$.

1.4.1 Excercise. Let X and Y be topological spaces. Show $X \times Y$ with the product topology is a topological space.

1.4.2 Excercise. Let X, Y, and Z be topological spaces. Show that a function $f : Z \to X \times Y$ is continuous if and only if the compositions of f with projections $\mathrm{pr}_1 : X \times Y \longrightarrow X$ and $\mathrm{pr}_2 : X \times Y \longrightarrow Y$ are both continuous.

1.5 Pull-backs

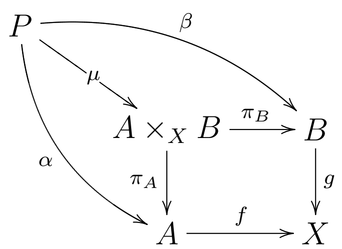
Let $f : A \longrightarrow X$ and $g : B \longrightarrow X$ be continuous maps between topological spaces. The subset $\{(a, b) \in A \times B \mid f(a) = g(b)\} \subset A \times B$, with the subspace topology, is called the pull-back of f and g. The pull-back is often denoted by $A \times_X B$ in which case we assume that the maps f and g are given.

1.5.1 Excercise. Let $f : A \longrightarrow X$ and $g : B \longrightarrow X$ be continuous maps.

(1) Show that the functions $\pi_A : A \times_X B \longrightarrow A$ and $\pi_B : A \times_X B \longrightarrow B$ which assign to (a, b) the elements a and b, respectively, are continuous.

(2) Let $f : A \longrightarrow X$ and $g : B \longrightarrow X$ be continuous maps between topological spaces. Show that for any two continuos maps $\alpha : P \longrightarrow A$ and $\beta : P \longrightarrow B$ for which $f\alpha = g\beta$, there is a unique continuous map $\mu : P \longrightarrow A_\times XB$ for which $\pi_A\mu = \alpha$ and $\pi_B\mu = \beta$.

(3) Show that the maps above fit into a commutative diagram:



We will sometime denote the pull-back of $f : A \longrightarrow X$ and $g : B \longrightarrow X$ as f^*B and the map $\pi_A : A \times_X B \longrightarrow B$ as $f^*g : f^*B \longrightarrow B$.

1.6 Quotients

Let X be a topological space, Y a set, and $f : X \longrightarrow Y$ a function of sets. The function f can be used to define a topology on Y. Let $\mathcal{U}$ be the collection of subset $U \subset Y$ for which $f^{-1}(U)$ is open in X. The collection $\mathcal{U}$ is called the quotient topology on Y induced by f. We will also call Y, with the quotient topology, a quotient space of X.

1.6.1 Excercise. Let X be a topological space. Show that Y with the quotient topology induced by $f : X \longrightarrow Y$ is a topological space and the function $f : X \longrightarrow Y$ is continuous.

An important property of a quotient space is that it is easy to verify if a function from such a space is continuous:

1.6.2 Proposition. *Let Y be the topological space given by the quotient topology induced by $f : X \longrightarrow Y$. A function $g : Y \to Z$ is continuous if and only if the composition $gf : X \longrightarrow Z$ is continuous.*

Proof. Composition of continuous functions is continuous (Exercise 1.1.6). Thus if g is continuous, then so is gf. This shows one implication.

Assume now that gf is continuous. We need to show that g is continuous. Let $U \subset Z$ be an open subset. As gf is continuous, the subset $f^{-1}(g^{-1}(U))$ is open in X. By definition of the quotient topology, $g^{-1}(U)$ is then open in Y and we can conclude that g is also continuous. □

1.7 Euclidean spaces

For $n > 0$, let $\mathbf{R}^n$ be the set of n-tuples of real numbers. If $n = 0$, we define $\mathbf{R}^0$ to be the one point set $\{0\}$. Recall that $|x| = \sqrt{x_1^2 + \cdots x_n^2}$, if $n > 0$ and $|0| = 0$ if $n = 0$. Let $a \in \mathbf{R}^n$ and $r \in \mathbf{R}$. The following subsets in $\mathbf{R}^n$ are called respectively the sphere, the disc, and the open ball with center in a and radius r:

$$S(a, r) = \{x \in \mathbf{R}^n \mid |x - a| = r\}$$

$$D(a, r) = \{x \in \mathbf{R}^n \mid |x - a| \leq r\}$$

$$B(a, r) = \{x \in \mathbf{R}^n \mid |x - a| < r\}$$

We can now define a topology on $\mathbf{R}^n$. A subset $U \subset \mathbf{R}^n$ is called open if, for any point $a \in U$, there is a number $\epsilon > 0$, such that the open ball $B(a, \epsilon)$ is included entirely in U. The collection of such open subsets of $\mathbf{R}^n$ is called the Euclidean topology and $\mathbf{R}^n$, with this choice of open subsets, is called the n-dimensional Euclidean space. We will not consider any other topology on $\mathbf{R}^n$. From now on the symbol $\mathbf{R}^n$ denotes the n-dimensional Euclidean space.

1.7.1 Excercise. Show that $\mathbf{R}^n$ with the above choice of open subsets is a Hausdorff topological space.

1.7.2 Excercise. Show that the product of Euclidean topologies on $\mathbf{R}^n \times \mathbf{R}^m$ is the same as the Euclidean topology on $\mathbf{R}^{n+m}$.

1.7.3 Excercise. Consider $\mathbf{R}^n$ as a subset of $\mathbf{R}^{n+m}$ consisting of $n+m$-tuples of real numbers whose last m-coordinates are 0. Show that the Euclidean topology on $\mathbf{R}^n$ is the same as the subspace topology of the Euclidean topology on $\mathbf{R}^{n+m}$.

We can now use the Euclidean space $\mathbf{R}^n$ to consider its subspaces. In this way we can get a lot of new examples of topological spaces. Here are some of the ones we will often use:

$$n\text{–dimensional disc:}\qquad D^n := \{x \in \mathbf{R}^n \mid |x| \leq 1\}$$

$$n\text{–dimensional open disc:}\qquad B^n := \{x \in \mathbf{R}^n \mid |x| < 1\}$$

$$(n-1)\text{–dimensional sphere:}\qquad S^{n-1} := \{x \in \mathbf{R}^n \mid |x| = 1\}$$

$$\text{unit interval:}\qquad I := \{x \in \mathbf{R} \mid 0 \leq x \leq 1\}$$

$$n\text{–dimensional simplex:}\qquad \Delta^n := \{x \in \mathbf{R}^{n+1} \mid x_0 + \cdots + x_n = 1\}$$

For example $D^0 = \mathbf{R}^0 = \Delta^0$ is the one point space. The space $S^0 \subset \mathbf{R}$ consists of two points $\{-1, 1\}$ and it is then isomorphic to the disjoint union $D^0 \coprod D^0$.

Let $0 \leq i \leq n$. A point in Δ^n whose all coordinates are 0 except the i-th coordinate, which has to be 1, is called the i-th vertex of Δ^n and is denoted by v_i.

1.7.4 Excercise. Let $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$ are subspaces. Show that a function $f : X \longrightarrow Y$ is continuous if and only if, for any $a \in X$ and any $\epsilon > 0$, there is $\delta > 0$ such that, when $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

1.7.5 Excercise. Show that, for any $a \in \mathbf{R}^n$ and any $r > 0$, the spaces $\mathbf{R}^n$ and $B(a, r)$ are isomorphic.

1.7.6 Excercise. Show that, for any $a, b \in \mathbf{R}^n$ and any $r, s > 0$, the spaces $D(a, r)$ and $D(b, s)$ are isomorphic.

1.7.7 Excercise. Show that D^n and Δ^n are isomorphic spaces.

1.8 Complex vector spaces

For $n > 0$, let $\mathbf{C}^n$ be the set of n-tuples of complex numbers. If $n = 0$, we define $\mathbf{C}^0$ to be the one point set $\{0\}$. Recall that for a complex number $z = a + ib$, $\overline{z} = a - ib$ and $|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}$. Recall also that, for $z \in \mathbf{C}^n$, $|z| = \sqrt{z_1\overline{z_1} + \cdots z_n\overline{z_n}}$, if $n > 0$ and $|0| = 0$ if $n = 0$.

We can define a topology on $\mathbf{C}^n$. A subset $U \subset \mathbf{C}^n$ is called open if, for any point $a \in U$, there is a number $\epsilon > 0$, such that $\{z \in \mathbf{C}^n \mid |z - a| < \epsilon\}$ is included entirely in U. We will not consider any other topology on $\mathbf{C}^n$.

1.8.1 Excercise. Show that $\mathbf{C}^n$ and $\mathbf{R}^{2n}$ are isomorphic topological spaces.

Let V be an n-dimensional $\mathbf{C}$-vector space. There is a $\mathbf{C}$-linear isomorphism $\phi : \mathbf{C}^n \longrightarrow V$. We can use this linear isomorphism to define a topology on V. We say that a subset $U \subset V$ is open if $\phi(U)$ is open in $\mathbf{C}^n$.

1.8.2 Excercise. Let V be an n-dimensional $\mathbf{C}$-vector space and $\phi : \mathbf{C}^n \longrightarrow V$ and $\psi : \mathbf{C}^n \longrightarrow V$ be $\mathbf{C}$-linear isomorphisms. Show that the topology on V induced by ϕ is the same as the topology induced by ψ.

Using the above exercise we can conclude that any finite dimensional $\mathbf{C}$-vector space has a natural topology. This is the only topology we are going to consider on such vector spaces.

1.9 Compact spaces

The aim of this section is to introduce an important class of topological spaces: compact spaces. Their key properties are: continuous bijections between them are isomorphisms and they admit a lot of continuous maps into an interval. We start with:

1.9.1 Definition.

(1) *A topological space X is called* **Hausdorff** *if, for any two distinct paints $x_1, x_2 \in X$, there are open subsets $x_1 \in U_1 \subset X$ and $x_2 \in U_2 \subset X$ whose intersection $U_1 \cap U_2$ is empty.*

(2) *A topological space X is* **compact** *if it is Hausdorff and, for any collection of open subsets $\{U_i \subset X\}_{i\in I}$ for which $\bigcup_{i\in I} U_i = X$, there is a finite sequence $i_1, i_2, \ldots, i_k$ such that:*

$$U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k} = X$$

The collection of open subsets $\{U_i \subset X\}_{i\in I}$ such that $\bigcup_{i\in I} U_i = X$ is called an open cover, or simply a cover, of X. A space X is compact if it is Hausdorff and any cover of X has a finite subcover.

1.9.2 Excercise. Let X and Y be spaces. The disjoint union $X \coprod Y$ is compact if and only if both X and Y are compact.

1.9.3 Excercise. Let $\{X_i\}_{i \in I}$ be a collection of compact spaces. Show that $\coprod_{i \in I} X_i$ is compact if and only if I is a finite set.

1.9.4 Excercise. Let X and Y be spaces. The product $X \times Y$ is compact if and only if both X and Y are compact.

1.9.5 Theorem. *A subspace X of the Euclidean space $\mathbf{R}^n$ is compact if and only if, as a subset of $\mathbf{R}^n$, it is closed and bounded (it lies in a ball $B(a, r)$ for some r).*

It follows from the above theorem that S^{n-1}, D^n, and Δ^n are compact spaces.

Here are some fundamental properties of compact spaces:

1.9.6 Proposition.

(1) Let X be a Hausdorff space. If a subspace $Y \subset X$ is compact, then it is a closed subset of X.

(2) Let X be a compact space. A subspace $Y \subset X$ is compact if and only if it is closed.

Proof. (1): We need to show that $X \setminus Y$ is open. For that it is enough to prove that, for any point $x \notin Y$, there is an open set $x \in U \subset X$ such that the intersection $U \cap Y$ is empty. Since X is Hausdorff, for any $y \in Y$, we can find two open subsets $x \in U_y \subset X$ and $y \in V_y \subset X$ such that $U_y \cap V_y = \emptyset$. It is then clear that $Y \subset \bigcup_{y \in Y} V_y$. Since Y is compact, we can then find finitely many points $y_1, \ldots, y_n$, such that $Y \subset V_{y_1} \cup \cdots \cup V_{y_n}$. It then follows that an open subset $U = U_{y_1} \cap \cdots \cap U_{y_n}$ has empty intersection with Y.
(2): If Y is compact then it is closed by statement (1).

Let Y be a closed subset of X and $Y = \bigcup_{i \in I}(U_i \cap Y)$, where U_i is open in X. Then $X = (X \setminus Y) \cup \bigcup_{i \in I} U_i$. Since X is compact we can find then a finite sequence $i_1, i_2, \ldots, i_k$ such that:

$$X = (X \setminus Y) \cup U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k}$$

It then follows that $Y = (U_{i_1} \cap Y) \cup (U_{i_2} \cap Y) \cup \cdots \cup (U_{i_k} \cap Y)$. To show that Y is compact, it remains to show that it is Hausdorff. Let y_1 and y_2 be two distinct points in Y. Since X is Hausdorff, there are two open subsets $y_1 \in U_1 \subset X$ and $y_2 \in U_2 \subset X$ whose intersection $U_1 \cap U_2$ is empty. It then follows that the intersection of $U_1 \cap Y$ and $U_2 \cap Y$ is also empty and Y is Hausdorff. □

An important property of compactness is that it is preserved by continuous functions:

1.9.7 Proposition.

(1) Let $f : X \longrightarrow Y$ be a continuous function and Y a Hausdorff space. If $D \subset X$ is compact, then $f(D)$ is compact and closed in Y.

(2) Assume that X is compact and Y Hausdorff. If $f : X \longrightarrow Y$ is a map which is an onto function, then Y is also compact.

(3) Assume that X is compact and Y is Hausdorff. If $f : X \longrightarrow Y$ is a continuos bijection (one to one and onto), then f is an isomorphism.

Proof. (1): Since Y is Hausdorff, then so is $f(D)$. Let $f(D) \subset \bigcup_{i\in I} U_i$. Since D is compact and $D \subset \bigcup_{i\in I} f^{-1}(U_i)$, there is a finite sequence $i_1, \ldots, i_k$ such that:

$$D \subset f^{-1}(U_{i_1}) \cup f^{-1}(U_{i_2}) \cup \cdots \cup f^{-1}(U_{i_k})$$

Consequently $f(D) \subset U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k}$. We can conclude that $f(D)$ is compact. By Proposition 1.9.6.(1), $f(D)$ is also closed in Y, as Y is assumed to be Hausdorff.
(2): This is a consequence of statement (1).
(3): Since f is a bijection, there is an inverse function $g : Y \longrightarrow X$, such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. We need to show that g is continuous. According to Exercise 1.1.5 it is enough to prove that $g^{-1}(D)$ is closed in Y, for any closed subset $D \subset X$. Note however that $g^{-1}(D) = f(D)$. Thus we need to show that $f(D)$ is closed. This follows from the following sequence of implications. Since D is closed and X is compact, D is also compact by Proposition 1.9.6.(2). As Y is Hausdorff, $f(D)$ is then compact by statement (1). We can then use again Proposition 1.9.6.(1) to conclude that $f(D)$ is closed. □

According to statement (3) of the above proposition, to show that compact spaces X and Y are isomorphic, it is enough to construct a continuous bijection $f : X \longrightarrow Y$. The inverse of f would be then necessarily continuous. This is one of the key advantages of compact spaces and we will use it often. Another property of compact spaces often used is:

1.9.8 Proposition. *Let X be a Hausdorff space and $Y \subset X$ and $Z \subset X$ be two disjoint ($Y \cap Z = \emptyset$) subspaces which are compact. Then there are open sets $Y \subset U \subset X$ and $Z \subset V \subset X$ such that $U \cap V = \emptyset$.*

Proof. Let us fix a point $y \in Y$. For any $z \in Z$ let us choose open subsets $y \in U_{y,z} \subset X$ and $z \in V_{y,z} \subset X$ such that $U_{y,z} \cap V_{y,z} = \emptyset$. This can be done since X is Hausdorff. Clearly $Z \subset \bigcup_{z \in Z} V_{y,z}$. Since Z is compact, there is a finite sequence $z_1, z_2, \ldots, z_k$ such that:

$$Z \subset V_{y,z_1} \cup V_{y,z_2} \cup \cdots \cup V_{y,z_k}$$

Define $U_y := U_{y,z_1} \cap Y_{y,z_2} \cap \cdots \cap U_{y,z_k}$ and $V_y := V_{y,z_1} \cup V_{y,z_2} \cup \cdots \cup V_{y,z_k}$. Then U_y and V_y are disjoint open subsets of X such that $y \in U_y$ and $Z \subset V_y$. Consider such subsets for all $y \in Y$. It is clear that $Y \subset \bigcup_{y \in Y} U_y$. Since Y is compact, there is a finite sequence $y_1, y_2, \ldots, y_n$ such that:

$$Y \subset U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_k}$$

Define $U := U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_k}$ and $V := V_{y_1} \cap V_{y_2} \cap \cdots \cap V_{y_k}$. The subsets U and V satisfy the requirements of the propositions. $\square$

1.9.9 Theorem. *Let X be a compact space.*

(1) If $A \subset X$, $B \subset X$ be non empty closed subsets, then there is a continuous function $f : X \longrightarrow I$ such that $f(A) = 0$ and $f(B) = 1$.

(2) If $A \subset X$ is a closed subset, then for any continuous function $f : A \longrightarrow \mathbf{C}^n$, there is a continuous function $g : X \longrightarrow \mathbf{C}^n$ such that $g(a) = f(a)$ for any $a \in A$.

(3) If $\{U_i \subset X\}_{1 \leq i \leq n}$ is an open covering of X, then there are functions $\{f_i : X \longrightarrow I\}_{1 \leq i \leq n}$ such that:

(a) $f_i(x) = 0$, if $x \notin U_i$

(b) $\Sigma_{i=1}^n f_i(x) = 1$ for any $x \in X$.