Lecture 3

Operations on complex vector spaces

1.16 Direct sums and homomorphisms

Let V and W be finite dimensional complex vector spaces. The direct sum $V \oplus W$ consists of pairs (v, w) of vectors $v \in V$ and $w \in W$. Addition and the action of $\mathbf{C}$ are defined coordinatewise

$$(v, w) + (v_1, w_1) = (v + v_1, w + w_1) \quad z(v, w) = (zv, zw)$$

The set $V \oplus W$ with these operations is a finite dimensional $\mathbf{C}$- vector space and hence a topological space.

If $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are basis of V and W, then:

$$\{(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)\}$$

is a base of $V \oplus W$. It follows that $\dim(V \oplus W) = \dim(V) + \dim(W)$.

The set of linear homomorphisms $\hom(V, W)$ has also a natural complex vector space structure given by the following operations. Let $f, g : V \longrightarrow W$ be linear homomorphisms.

$$(f + g)(v) := f(v) + g(v) \qquad (zf)(v) := z(f(v))$$

1.16.1 Excercise. Let V and W be complex vector spaces.

(1) Show that, for any vector space U,

$$\hom(V \oplus W, U) \text{ and } \hom(V, U) \oplus \hom(W, U)$$

are isomorphic vector spaces.

(2) If T is a complex vector space such that, for any U, $\hom(T, U)$ and $\hom(V, U) \oplus \hom(W, U)$ are isomorphic, then T is isomorphic to $V \oplus W$.

(3) Show that $V \oplus W$ and $W \oplus V$ are isomorphic vector spaces.

Let us choose basis $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ in V and W. Let $\delta_{ij} : V \longrightarrow W$ be the unique homomorphism such that:

$$\delta_{ij}(v_s) = \begin{cases} w_i & \text{if } s = j \\ 0 & \text{if } s \neq j \end{cases}$$

The set $\{\delta_{ij}\}_{1\leq i\leq m, 1\leq j\leq n}$ is a base for $\hom(V,W)$. Consequently:

$$\dim(\hom(V,W)) = \dim(V)\dim(W)$$

Furthermore, if $f : V \longrightarrow W$ is a linear function, then it can be written as a linear combination:

$$f : \Sigma_{i,j} c_{ij}\delta_{ij}$$

Thus $f(v_j) = \Sigma_{i=1}^{m} c_{ij} w_i$. This means that $[c_{ij}]_{1\leq i\leq m, 1\leq j\leq n}$ is the standard matrix associated to f with respect to the chosen basis. The association $f \mapsto [c_{ij}]$ is a linear isomorphism between $\hom(V,W)$ and $m\times n$ complex matrices. Such matrices can be identified with $\mathbf{C}^{nm}$.

Let $f : V \longrightarrow V$ be a linear function. Let us choose a base $\{v_1,\ldots,v_n\}$ in V. Let $[c_{ij}]_{1\leq i\leq n, 1\leq j\leq n}$ be the matrix associated to f with respect to the chosen base. We can use this matrix to define:

$$\det(f) := \det[c_{ij}]$$

1.16.2 Excercise. (1) Show that, for a linear function $f : V \longrightarrow V$, $\det(f)$ does not depend on the choice of a base in V.

(2) Show that $\det : \hom(V,V) \longrightarrow \mathbf{C}$ is continuous.

(3) Show that the composition function:

$$\hom(V,W)\times\hom(W,U) \ni (f,g) \mapsto gf \in \hom(V,U)$$

is continuous.

For an n-dimensional complex vector space V, we define $GL(V)$ to be the subset of $\hom(V,V)$ that consists of these linear functions $f : V \longrightarrow V$ which are isomorphisms. We think about $GL(V)$ as a topological space with the topology given by the subspace topology of $\hom(V,V)$. If we choose a base in V and identify $\hom(V,V)$ with $n\times n$ complex matrices, then $GL(V)$ can be identified with these matrices whose determinant is not 0. Thus the determinant induces a continuous function

$$\det : GL(V) \longrightarrow \mathbf{C}^*$$

where $\mathbf{C}^*$ is the subspace of non-zero complex numbers in $\mathbf{C}$.

1.16.3 Excercise. Let V be a finite dimensional complex vector space.

(1) Show that $GL(V) \ni f \mapsto f^{-1} \in GL(V)$ is a continuous function.

(2) Show that $\pi_0(\det) : \pi_0(GL(V)) \longrightarrow \pi_0(\mathbf{C}^*)$ is a bijection.

(3) Show that $GL(V)$ is a path connected space.

1.17 Tensor products

Let V and W be complex vector spaces. Consider a complex vector space T whose base is given by all the pairs of vectors (v, w) where $v \in V$ and $w \in W$. Thus elements in T are given by finite linear combinations

$$z_1(v_1, w_1) + \cdots + z_n(v_n, w_n)$$

where z_i's are complex numbers. Let $U \subset T$ be a vector subspace generated by:

$$(zv, w) - (v, zw) \qquad z(v, w) - (zv, w)$$

$$(v_1 + v_2, w) - (v_1, w) - (v_2, w) \qquad (v, w_1 + w_1) - (v, w_1) - (v, w_2)$$

for all vectors v, v_1, v_2 in V, w, w_1, w_2 in W and all complex numbers z.

Define the tensor product of V and W to be the quotient vector space $V \otimes W := T/U$. Define further a function $\mu : V \times W \longrightarrow V \otimes W$ by:

$$\mu(v, w) := (v, w)U$$

1.17.1 Excercise. Show that μ has the following properties:

$$\mu(z_1v_1 + z_2v_2, w) = z_1\mu(v_1, w) + z_2\mu(v_2, w)$$

$$\mu(v, z_1w_1 + z_2w_2) = z_1\mu(v, w_1) + z_2\mu(v, w_2)$$

We can use the above properties of μ to define so called bilinear functions. We say that a function $f : V \times W \longrightarrow U$ is bilinear if:

$$f(z_1v_1 + z_2v_2, w) = z_1f(v_1, w) + z_2f(v_2, w)$$

$$f(v, z_1w_1 + z_2w_2) = z_1f(v, w_1) + z_2f(v, w_2)$$

for any vectors v, v_1, and v_2 in V, w, w_1, and w_2 in W and any complex numbers z_1 and z_2. We use the symbol $B(V, W|U)$ to denote the set of bilinear functions $f : V \times W \longrightarrow U$. Note that if $f, g : V \times W \longrightarrow U$ are bilinear, then so are $f + g$ and zf for any complex number z. These operations define a complex vector space structure on $B(V, W|U)$.

1.17.2 Excercise. Let V and W be complex vector spaces.

(1) Show that, for any vector space U, $B(V, W|U)$ and $\hom(V, \hom(W, U))$ are isomorphic vector spaces.

(2) Show that for any bilinear map $f : V \times W \longrightarrow U$, there is a unique linear map $g : V \otimes W \longrightarrow U$ for which $g\mu = f$.

(3) Show that, for any vector space U, $B(V, W|U)$ and $\hom(V \otimes W, U)$ are isomorphic vector spaces.

(4) Show that if T is a vector space such that, for any U, $B(V, W|U)$ and $\hom(T, U)$ are isomorphic, then T and $V \otimes W$ are isomorphic too.

(5) Show that $V \otimes W$ and $W \otimes V$ are isomorphic.

(6) Show that $(V \oplus W) \otimes U$ and $(V \otimes U) \oplus (W \otimes U)$ are isomorphic.

1.18 K-theory

Assume that we are given a set T, two elements $0, 1 \in T$, and two operations:

$$+ : T \times T \longrightarrow T \qquad \otimes : T \times T \longrightarrow T$$

with the following properties:

(1) $(a + b) + c = a + (b + c)$

(2) $a + b = b + a$

(3) $0 + a = a$

(4) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$

(5) $a \otimes b = b \otimes a$

(6) $1 \otimes a = a$

(7) $a \otimes (b + c) = (a \otimes b) + (a \otimes c)$

The set T with the above operations is not a commutative ring as the addition may not have inverses. Our first goal is to transform T into a commutative ring by adding additive inverses.

Consider the set of pairs $T \times T$ and the following relation on it:

$$(a, b) \simeq (c, d) \text{ if } a + d = b + c$$

1.18.1 Excercise. Show that $\simeq$ is an equivalence relation on $T \times T$.

We are going to use the symbol $\widehat{T}$ to denote the set of equivalence classes of the relation $\simeq$ on $T \times T$. For $(a, b) \in T \times T$, we are going to denote by $a - b$ the element in $\widehat{T}$ which is given by the equivalence class represented by the pair (a, b). Thus $a - b = c - d$ in $\widehat{T}$ if and only if $a + d = b + c$ in T.

We define further:

$$1 := 1 - 0 \qquad 0 := 0 - 0$$

$$(a - b) + (c - d) := (a + c) - (b + d)$$

$$(a - b) \otimes (c - d) := (a \otimes c + b \otimes d) - (a \otimes d + b \otimes c)$$

1.18.2 Excercise. Show that:

(1) the operations $+$ and $\otimes$ are well define on $\hat{T}$.

(2) $a - a = 0$,

(3) $(a - b) + (c - d) = (c - d) + (a - b)$

(4) $(a - b) + (b - a) = 0$

(5) the set $\widehat{T}$ with 0 as the zero element, 1 as the unit element, and the operations $+$ and $\otimes$ is a commutative ring.

(6) Show that the function $\mu : T \longrightarrow \widehat{T}$ which assigns to $a \in T$ the element $a - 0 \in \widehat{T}$ satisfies the following properties:

$$\mu(0) = 0,\ \mu(1) = 1,\ \mu(a + b) = \mu(a) + \mu(b),\ \mu(a \otimes b) = \mu(a) \otimes \mu(b)$$

(7) Show that for any other function $\alpha : T \longrightarrow R$ from T to a commutative ring R that satisfies the analogous to the above properties of μ, there is a unique ring homomorphism $\beta : \widehat{T} \longrightarrow R$ for which $\beta\mu = \alpha$.

According to the above exercises, with T we associated a commutative ring $\widehat{T}$ and a comparison function $\mu : T \longrightarrow \widehat{T}$.

1.18.3 Example. Consider the natural numbers $\mathbf{N}$ with the usual addition and multiplication. Then $\widehat{\mathbf{N}}$ can be identified with the ring of integers $\mathbf{Z}$ in such way that the function μ is given by the usual inclusion $\mathbf{N} \subset \mathbf{Z}$.

1.18.4 Example. Let T be the set of isomorphism classes of finite dimensional complex vector spaces. For two vector spaces V and W define $V + W$ to be the direct sum $V \oplus W$ and $V \otimes W$ to be the tensor product of V and W. We also take 0 to be the trivial (0-dimensional) complex vector space and 1 to be the 1-dimensional complex vector space. The set T with this choice of elements and operations satisfies the required properties. The commutative ring $\widehat{T}$ is denoted by $K(D^0)$. Note that $K(D^0)$ is isomorphic to $\mathbf{Z}$.

1.18.5 Excercise. Give an example of T for which $\mu : T \longrightarrow \widehat{T}$ is not an inclusion.