Lecture 5

Constructing new vector bundles

1.21 Pull-back

Let $p : E \longrightarrow X$ be a vector bundle and $f_0 : Y \longrightarrow X$ be a continuous map. Define:

- $F := \{(y, e) \in Y \times E \mid f_0(y) = p(e)\}$ with the subspace topology in $Y \times E$;
- $f_0^* p : F \longrightarrow Y$ to be the composition of the inclusion $F \subset Y \times E$ and the projection $\mathrm{pr}_Y : Y \times E \longrightarrow Y$;
- $f : F \longrightarrow E$ to be the composition of the inclusion $F \subset Y \times E$ and the projection $\mathrm{pr}_E : Y \times E \longrightarrow E$.

As compositions of continuous maps, the functions $f_0^* p$ and f are continuous.

1.21.1 Excercise. Show that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "f_0^* p"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f_0"] & X \end{tikzcd}$$

Note that for $y \in Y$, $(f_0^* p)^{-1}(y) = \{y\} \times p^{-1}(f_0(y))$. Thus we can identify $(f_0^* p)^{-1}(y)$ with $p^{-1}(f_0(y))$. Via this identification, $(f_0^* p)^{-1}(y)$ becomes a complex vector space.

1.21.2 Proposition. *The map $f_0^* p : F \longrightarrow Y$ together with the complex vector space structures on $(f_0^* p)^{-1}(y)$, for $y \in Y$ is a complex vector bundle.*

Proof. Let $U \subset X$ be an open subset over which p is trivial. We can then choose inverse isomorphism ϕ and ϕ^{-1} for which the following diagram commutes:

$$\begin{tikzcd} U \times \mathbf{C}^n \arrow[r, "\phi"] \arrow[dr, "\mathrm{pr}"'] & p^{-1}(U) \arrow[r, "\phi^{-1}"] \arrow[d, "p"] & U \times \mathbf{C}^n \arrow[dl, "\mathrm{pr}"] \\ & U & \end{tikzcd}$$

Let $V = f_0^{-1}(U)$. We claim that $f_0^* p$ is trivial over V. Define:

- $\psi : V \times \mathbf{C}^n \longrightarrow (f_0^* p)^{-1}(V) \subset V \times p^{-1}(U), \quad \psi(y, v) := (y, \phi(f_0(y), v))$
- $\psi^{-1} : (f_0^* p)^{-1}(V) \longrightarrow V \times \mathbf{C}^n, \quad \psi^{-1}(e) := (f_0^* p(y), \phi^{-1} f(e))$

From this definition it is clear that ψ and ψ^{-1} are continuous functions and that they are inverse to each other. $\square$

The vector bundle $f_0^* p : F \longrightarrow Y$ is called the **pull-back** of p along $f_0 : Y \longrightarrow X$.

1.22 External product

Let $p : E \longrightarrow X$ and $q : F \longrightarrow Y$ be vector bundles. Then the product map $p \times q : E \times F \longrightarrow X \times Y$ is also a vector bundle with the vector space structure on $(p \times q)^{-1}(x, y) = p^{-1}(x) \times q^{-1}(y)$ given by the product structure. This vector bundle is called the **external product** of p and q.

To see the triviality of $p \times q$ assume that $\{U_i\}_{i \in I}$ is an open cover of X over which p is trivial and $\{V_j\}_{j \in J}$ is an open cover of Y over which q is trivial. Consider the cover $\{U_i \times V_j\}_{i \in I, j \in J}$ of the product $X \times Y$. We claim that $p \times q$ is trivial over $U_i \times V_j$ for any $i \in I$ and $j \in J$.

1.22.1 Excercise. Show the triviality of $p \times q$ over $U_i \times V_j$.

1.23 Direct sum

Let $p : E \longrightarrow X$ and $q : F \longrightarrow X$ be vector bundles over the same base X. Define:

- $E \oplus F := \coprod_{x \in X} p^{-1}(x) \oplus q^{-1}(x)$;
- $p \oplus q : E \oplus F \longrightarrow X$ to be a function that maps elements in $p^{-1}(x) \oplus q^{-1}(x)$ to x. Thus $(p \oplus q)^{-1}(x) = p^{-1}(x) \oplus q^{-1}(x)$.
- The $\mathbf{C}$-vector space structure on $(p \oplus q)^{-1}(x)$ to be the direct product of the vector spaces $p^{-1}(x)$ and $q^{-1}(x)$.

To define the topology on $E \oplus F$ we need to choose trivializations of p and q. Assume that p is trivial over $\{U_i\}_{i \in I}$ and q is trivial over $\{V_j\}_{j \in J}$. Let $\phi_i : U_i \times \mathbf{C}^n \longrightarrow p^{-1}(U_i)$ and $\psi_j : V_j \times \mathbf{C}^m \longrightarrow q^{-1}(V_j)$ be isomorphism corresponding to the open subsets U_i and V_j. Consider the open cover $\{W_{i,j} := U_i \cap V_j\}_{i \in I, j \in J}$ of X and define:

- $\mu_{i,j} : W_{i,j} \times (\mathbf{C}^n \oplus \mathbf{C}^m) \longrightarrow (p \oplus q)^{-1}(W_{i,j}) = \coprod_{x \in W_{i,j}} p^{-1}(x) \oplus q^{-1}(x)$ to be given by:

$$\mu_{i,j}(x, v, w) := (\phi(x, v), \psi(x, w))$$

- A subset $T \subset E \oplus F$ to be open if and only if $\mu_{i,j}^{-1}(T \cap (p \oplus q)^{-1}(W_{i,j}))$ is open in $W_{i,j} \times (\mathbf{C}^n \oplus \mathbf{C}^m)$ for any $i \in I$ and $j \in J$.

1.23.1 Excercise. (1) Show that the above definition of open subsets in $E \oplus F$ defines a topology on $E \oplus F$ for which $p \oplus q : E \oplus F \longrightarrow X$ is a continuous function.

(2) Show that this topology does not depend on the choice of the trivializations ϕ_i and ψ_j.

(3) Show that $\mu_{i,j}$ are isomorphism.

(4) Conclude that $p \oplus q : E \oplus F \longrightarrow X$ is a vector bundle.

The map $p \oplus q : E \oplus F \longrightarrow X$ together with the vector space structures on its fibers $(p \oplus q)^{-1}(x)$, for $x \in X$, is called the **direct sum** of p and q.

1.23.2 Excercise. Let $\Delta : X \longrightarrow X \times X$ be the diagonal map that maps x to $\Delta(x) = (x, x)$. Show that for any complex vector bundles $p : E \longrightarrow X$ and $q : F \longrightarrow X$, the pull-back of the external product $\Delta^*(p \times q)$ is isomorphic to the direct sum $p \oplus q$.

1.24 Tensor product

We follows the same procedure as in the case of the direct sum. Let $p : E \longrightarrow X$ and $q : F \longrightarrow X$ be vector bundles over the same base X. Define:

- $E \otimes F := \coprod_{x \in X} p^{-1}(x) \otimes q^{-1}(x)$;
- $p \otimes q : E \otimes F \longrightarrow X$ to be a function that maps elements in $p^{-1}(x) \otimes q^{-1}(x)$ to x. Thus $(p \otimes q)^{-1}(x) = p^{-1}(x) \otimes q^{-1}(x)$.
- The $\mathbf{C}$-vector space structure on $(p \otimes q)^{-1}(x)$ to be the tensor product of the vector spaces $p^{-1}(x)$ and $q^{-1}(x)$.

To define the topology on $E \otimes F$ we need to choose trivializations of p and q. Assume that p is trivial over $\{U_i\}_{i \in I}$ and q is trivial over $\{V_j\}_{j \in J}$. Let $\phi_i : U_i \times \mathbf{C}^n \longrightarrow p^{-1}(U_i)$ and $\psi_j : V_j \times \mathbf{C}^m \longrightarrow q^{-1}(V_j)$ be isomorphism corresponding to the open subsets U_i and V_j. Consider the open cover $\{W_{i,j} := U_i \cap V_j\}_{i \in I, j \in J}$ of X and define:

- $\mu_{i,j} : W_{i,j} \times (\mathbf{C}^n \otimes \mathbf{C}^m) \longrightarrow (p \otimes q)^{-1}(W_{i,j}) = \coprod_{x \in W_{i,j}} p^{-1}(x) \otimes q^{-1}(x)$ to be given by:

$$\mu_{i,j}(x, v \otimes w) := (\phi(x,v) \otimes \psi(x,w))$$

- A subset $T \subset E \otimes F$ to be open if and only if $\mu_{i,j}^{-1}(T \cap (p \otimes q)^{-1}(W_{i,j}))$ is open in $W_{i,j} \times (\mathbf{C}^n \otimes \mathbf{C}^m)$ for any $i \in I$ and $j \in J$.

1.24.1 Excercise. (1) Show that the above definition of open subsets in $E \otimes F$ defines a topology on $E \otimes F$ for which $p \otimes q : E \otimes F \longrightarrow X$ is a continuous function.

(2) Show that this topology does not depend on the choice of the trivializations ϕ_i and ψ_j.

(3) Show that $\mu_{i,j}$ are isomorphism.

(4) Conclude that $p \otimes q : E \otimes F \longrightarrow X$ is a vector bundle.

The map $p \otimes q : E \otimes F \longrightarrow X$ together with the vector space structures on its fibers $(p \otimes q)^{-1}(x)$, for $x \in X$, is called the **tensor product** of p and q.

1.25 Hom

We follows the same procedure as in the case of the direct sum and tensor product. Let $p : E \longrightarrow X$ and $q : F \longrightarrow X$ be vector bundles over the same base X. Define:

- $\mathrm{Hom}(E,F) := \coprod_{x \in X} \mathrm{hom}(p^{-1}(x), q^{-1}(x))$;
- $\mathrm{Hom}(p,q) : \mathrm{Hom}(E,F) \longrightarrow X$ to be a function that maps elements in $f \in \mathrm{hom}(p^{-1}(x), q^{-1}(x))$ to x. Thus:

$$\mathrm{Hom}(p,q)^{-1}(x) = \mathrm{hom}(p^{-1}(x), q^{-1}(x))$$

- The $\mathbf{C}$-vector space structure on $\mathrm{Hom}(p,q)^{-1}(x)$ to be the standard complex vector space structure on the set of linear homomorphisms $\mathrm{hom}(p^{-1}(x), q^{-1}(x))$.

To define the topology on $\mathrm{Hom}(E,F)$ we need to choose trivializations of p and q. Assume that p is trivial over $\{U_i\}_{i \in I}$ and q is trivial over $\{V_j\}_{j \in J}$. Let $\phi_i : U_i \times \mathbf{C}^n \longrightarrow p^{-1}(U_i)$ and $\psi_j : V_j \times \mathbf{C}^m \longrightarrow q^{-1}(V_j)$ be isomorphism corresponding to the open subsets U_i and V_j. Consider the open cover $\{W_{i,j} := U_i \cap V_j\}_{i \in I, j \in J}$ of X and define:

- $\mu_{i,j} : W_{i,j} \times \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^m) \longrightarrow \mathrm{Hom}(p,q)^{-1}(W_{i,j}) = \coprod_{x \in W_{i,j}} \mathrm{hom}(p^{-1}(x), q^{-1}(x))$ to be given by:

$$\mu_{i,j}(x, f) := (\phi(x, -) f \psi(x, -)^{-1})$$

- A subset $T \subset \mathrm{Hom}(E, F)$ to be open if and only if $\mu_{i,j}^{-1}(T \cap \mathrm{Hom}(p,q)^{-1}(W_{i,j}))$ is open in $W_{i,j} \times \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^m)$ for any $i \in I$ and $j \in J$.

1.25.1 Excercise. (1) Show that the above definition of open subsets in $\mathrm{Hom}(E, F)$ defines a topology on $\mathrm{Hom}(E, F)$ for which $\mathrm{Hom}(p, q) : \mathrm{Hom}(E, F) \longrightarrow X$ is a continuous function.

(2) Show that this topology does not depend on the choice of the trivializations ϕ_i and ψ_j.

(3) Show that $\mu_{i,j}$ are isomorphism.

(4) Conclude that $\mathrm{Hom}(p, q) : \mathrm{Hom}(E, F) \longrightarrow X$ is a vector bundle.

The map $\mathrm{Hom}(p, q) : \mathrm{Hom}(E, F) \longrightarrow X$ together with the vector space structures on its fibers $\mathrm{Hom}(p, q)^{-1}(x)$, for $x \in X$, is called the **Hom bundle** between p and q.

1.26 The K-theory of a topological space

Let n be a natural number. We are going to denote by $\mathbf{n}$ the product vector bundle: $\mathrm{pr} : X \times \mathbf{C}^n \longrightarrow X$ with the standard vector space structure on the fiber $\mathbf{C}^n$. For example $\mathbf{0}$ is the vector bundle $\mathrm{id} : X \longrightarrow X$ with the 0-dimensional vector space structure on any fiber of $\mathrm{id} : X \longrightarrow X$, and $\mathbf{1}$ is the vector bundle $\mathrm{pr} : X \times \mathbf{C}^1 \longrightarrow X$ with the 1-dimensional vector space structure on any fiber $\mathbf{C}^1$.

1.26.1 Excercise. Show that:

(1) $\mathbf{n} \oplus \mathbf{m}$ and $\mathbf{n} + \mathbf{m}$ are isomorphic.

(2) $\mathbf{n} \otimes \mathbf{m}$ and $\mathbf{nm}$ are isomorphic.

1.26.2 Proposition. *Let $p : E \longrightarrow X$, $q : F \longrightarrow X$, and $r : G \longrightarrow X$ be vector bundles. Then*

(1) $p \oplus q$ is isomorphic to $q \oplus p$;

(2) $(p \oplus q) \oplus r$ is isomorphic to $q \oplus (p \oplus r)$;

(3) $p \oplus \mathbf{0}$ *is isomorphic to* p

(4) $p \otimes q$ *is isomorphic to* $q \otimes p$

(5) $(p \otimes q) \otimes r$ *is isomorphic to* $q \otimes (p \otimes r)$

(6) $p \otimes \mathbf{1}$ *is isomorphic to* p

(7) $p \otimes q \oplus r$ *is isomorphic to* $(p \otimes q) \oplus (p \otimes r)$

Proof. One uses the same argument to prove all these statements. We are going to illustrate only how to show (4). Adjustment of this argument to other cases is left as an exercise. Let $f : E \otimes F \longrightarrow F \otimes E$ to be given by the following formula:

$$E \otimes F = \coprod_{x \in X} p^{-1}(x) \otimes q^{-1}(x) \ni v \times w \mapsto w \otimes v \in \coprod_{x \in X} q^{-1}(x) \otimes p^{-1}(x) = F \times E$$

Using 1.20.1, it is straightforward to see that f is a vector bundle isomorphism. $\square$

We define $\mathrm{Vect}(X)$ to be the set of isomorphism classes of complex vector bundles over X. The above proposition states that this set with the operations $\oplus$, $\otimes$, and chosen elements $\mathbf{0}$ and $\mathbf{1}$ satisfies the properties given in Sections 1.18.

1.26.3 Definition. $K(X) := \widehat{\mathrm{Vect}(X)}$ *(see Sections 1.18).*

Thus $K(X)$ is a commutative ring whose elements are denoted by $p - q$ where $p : E \longrightarrow X$ and $q : F \longrightarrow X$ are complex vector bundles over X. Two such elements $p - q$ and $p' - q'$ are equal if and only if there is a vector bundle $r : G \longrightarrow X$ such that $p \oplus q' \oplus r$ and $p' \oplus q \oplus t$ are isomorphic.

We are going to denote the element $p - \mathbf{0}$ simply by p. In the ring $K(X)$ the addition is given by $(p - q) \oplus (p' - q') = (p \oplus p') - (q \oplus q')$. The unit element for this addition is given by $\mathbf{0}$. The multiplication in $K(X)$ is given by $(p - q) \otimes (p' - q') = ((p \otimes p') \oplus (q \otimes q')) - ((p \otimes q') \oplus (q \otimes p'))$. The unit for this multiplication is given by $\mathbf{1}$.

1.26.4 Example. $\mathrm{Vect}(D^0)$ is isomorphic to the set of natural numbers $\mathbf{N}$ with the usual addition and multiplication. In this case $K(D^0)$ is the ring of integers $\mathbf{Z}$.

1.27 Sections of vector bundles

Let $p : E \longrightarrow X$ be a vector bundle and $Y \subset X$ be a subspace. A section of p over Y is by definition a continuous function $s : Y \longrightarrow E$ such that $ps = \mathrm{id}_Y$. We use the symbol $\Gamma(p, Y)$ to denote the set of all sections of p over Y.

Recall that $p^{-1}(x)$ is a complex vector space for any $x \in X$. Using this structure we can define the complex space structure on $\Gamma(p, Y)$. Let $z \in \mathbf{C}$ be a complex number and $s, t \in \Gamma(p, Y)$ be sections of p over $Y \subset X$. Define:

- $zs : Y \longrightarrow E$ to be the function that assigns to an element y the element $zs(y) \in p^{-1}(y)$, where the multiplication by a complex number is given by the complex vector space structure on $p^{-1}(y)$.
- $s + t : Y \longrightarrow E$ to be be the function that assigns to an element y the element $(s + t)(y) = s(y) + t(y) \in p^{-1}(y)$, where the addition is given by the complex vector space structure on $p^{-1}(y)$.

1.27.1 Excercise. (1) Show that $zs : Y \longrightarrow E$ and $s + t : Y \longrightarrow E$ are continuous functions. Conclude that these functions belong to $\Gamma(p, Y)$.

(2) Show that the above operations define a complex vector space structure on $\Gamma(p, Y)$.

1.27.2 Excercise. Let $p : E \longrightarrow X$ and $q : F \longrightarrow X$ be vector bundles. For any section $s \in \Gamma(\mathrm{Hom}(p, q), X)$, the elements $s(x) \in \mathrm{Hom}(p, q)^{-1}(x) = \hom(p^{-1}(x), q^{-1}(x))$ is simply a homomorphism $s(x) : p^{-1}(x) \longrightarrow q^{-1}(x)$.

(1) Show that the function $\alpha(s) : E \longrightarrow F$ that assigns to a point $e \in p^{-1}(x)$ the point $s(x)(e) \in q^{-1}(x)$ is continuous.

(2) Conclude that there is a bijection between the set $\Gamma(\mathrm{Hom}(p, q), X)$ and the set of bundle maps between E and F.