Lecture 6

1.28 Vector bundles and maps

Let $f : Y \longrightarrow X$ be a continuous map. The pull-back operation can be used to define a function:

$$\mathrm{Vect}(X) \ni (p : E \longrightarrow X) \mapsto (f^*p : F \longrightarrow Y) \in \mathrm{Vect}(Y)$$

which we are going to denote by the symbol $\mathrm{Vect}(f) : \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(Y)$.

1.28.1 Excercise. Let $p : E \longrightarrow X$ and $q : F \longrightarrow X$ be complex vector bundles and $f : Y \longrightarrow X$ be a continuous map. Show that:

(1) $\mathrm{Vect}(f)(p \oplus q) = \mathrm{Vect}(f)(p) \oplus \mathrm{Vect}(f)(q)$, i.e, $f^*(p \oplus q)$ and $f^*(p) \oplus f^*(q)$ are isomorphic.

(2) $\mathrm{Vect}(f)(\mathbf{n}) = \mathbf{n}$, for $n \geq 0$.

(3) $\mathrm{Vect}(f)(p \otimes q) = \mathrm{Vect}(f)(p) \otimes \mathrm{Vect}(f)(q)$, i.e, $f^*(p \otimes q)$ and $f^*(p) \otimes f^*(q)$ are isomorphic.

It follows from the above exercise that $\mathrm{Vect}(f)$ induces a unique ring homomorphism $K(f) : K(X) \longrightarrow K(Y)$ for which the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Vect}(X) & \longrightarrow & \widehat{\mathrm{Vect}(X)} & = & K(X) \\ \downarrow{\scriptstyle \mathrm{Vect}(f)} & & & & \downarrow{\scriptstyle K(f)} \\ \mathrm{Vect}(Y) & \longrightarrow & \widehat{\mathrm{Vect}(Y)} & = & K(Y) \end{array}$$

The homomorphism $K(f) : K(X) \longrightarrow K(Y)$ takes an element $p - q$ in $K(X)$ to an element $f^*p - f^*q$ in $K(Y)$.

1.28.2 Excercise. Let $g : Z \longrightarrow Y$ and $f : Y \longrightarrow X$ be continuous maps. Show:

(1) $\mathrm{Vect}(fg) = \mathrm{Vect}(g)\mathrm{Vect}(f)$;

(2) $\mathrm{Vect}(\mathrm{id}) = \mathrm{id}$;

(3) $K(fg) = K(g)K(f)$;

(4) $K(\mathrm{id}) = \mathrm{id}$;

(5) Conclude that if $f : Y \longrightarrow X$ is an isomorphism, then so are $\mathrm{Vect}(f)$ and $K(f)$.

1.29 Homotopical properties I

It turns out that $K(-)$ not only transforms isomorphisms into isomorphisms, but also homotopy equivalences.

1.29.1 Theorem. *Let Y be a compact space and $f, g : Y \longrightarrow X$ be continuous maps. If f and g are homotopic, then* $\mathrm{Vect}(f) = \mathrm{Vect}(g)$.

For the proof we need two lemmas:

1.29.2 Lemma. *Let $p : E \longrightarrow X$ be a vector bundle. Assume that X is compact and $Y \subset X$ is a closed subset. Then any section $s \in \Gamma(p, Y)$ can be extended to a section $\hat{s} \in \Gamma(p, X)$, i.e., the restriction function $\Gamma(p, X) \longrightarrow \Gamma(p, Y)$ is onto.*

Proof. **Step 1.** Assume first that p is the product bundle $\mathbf{n}$ given by the projection $X \times \mathbf{C}^n \longrightarrow X$. In this case consider the composition of the section $s : Y \longrightarrow Y \times \mathbf{C}^n$ and the projection $Y \times \mathbf{C}^n \longrightarrow \mathbf{C}^n$ which we denote by f. Since X is compact, according to 1.9.9.(2), there is a function $g : X \longrightarrow \mathbf{C}^n$ for which $g(y) = f(y)$ if $y \in Y$. Define $\hat{s} : X \longrightarrow X \times \mathbf{C}^n$ by the formula $\hat{s}(x) = (x, g(x))$. It is then clear that $\hat{s}$ is the desired section.

Step 2. For any $x \in X$, let us choose an open subset $x \in U_x \subset X$ over which p is trivial. Let $\phi_x : U_x \times \mathbf{C}^n \longrightarrow p^{-1}(U_x)$ be a trivialization of p over U_x. Let $h_x : X \longrightarrow I$ be a function such that $h_x(x) = 0$ and $h_x(X \setminus U_x) = 1$. Such a function exists by 1.9.9.(1). Let $V_x = h_x^{-1}([0, 1/2))$ and $\overline{U_x} = h_x^{-1}([0, 1/2])$. Note that $\overline{V_x}$ is a closed subset of X over which p is the product bundle $\mathbf{n}$. According to step 1, there is then a section $s_x \in \Gamma(p, \overline{V_x})$ such that $s_x(y) = s(y)$ for any $y \in Y \cap \overline{V_x}$.

Since X is compact we can choose a sequence $x_1, \ldots, x_k$ of points in X for which $X = \bigcup_{i=1}^k V_{x_i}$. We use the same symbol $s_{x_i} \in \Gamma(p, V_{x_i})$ to denote the restriction of s_{x_i} to $V_{x_i} \subset \overline{V_{x_i}}$.

Let $\{f_i : X \longrightarrow I\}_{1 \le i \le k}$ be a sequence of maps such that $f_i(x) = 0$ if $x \notin V_{x_i}$ and $\Sigma_{i=1}^k f_i(x) = 1$ for any $x \in X$. Such maps exist by 1.9.9.(3). Note that, for $1 \le i \le k$, the following formula describes a section in $\Gamma(p, X)$ which we denote by $s_i \in \Gamma(p, X)$:

$$X \ni x \mapsto \begin{cases} f_i(x) s_{x_i}(x) & \text{if } x \in V_{x_i} \\ 0 & \text{if } x \notin V_{x_i} \end{cases}$$

Define the section $(\hat{s} : X \longrightarrow E) \in \Gamma(X, p)$ by the formula:

$$\hat{s}(x) := \Sigma_{i=1}^k s_i$$

Note that if $x \in Y$, then $\hat{s}(x) := \Sigma_{i=1}^k s_i(x) = s(x)$. $\square$

If $p : E \longrightarrow X$ is a vector bundle and $Y \subset X$ is a subspace. The pull-back of p along the inclusion $Y \subset X$ is denoted by $p|_Y : E|_Y \longrightarrow Y$.

1.29.3 Lemma. *Let $p : E \longrightarrow X$ and $q : F \longrightarrow X$ be vector bundles. Assume that X is compact and $Y \subset X$ is a closed subset. If $p|_Y$ and $q|_Y$ are isomorphic, then there is an open subset $Y \subset U \subset X$ for which $p|_U$ and $q|_U$ are isomorphic.*

Proof. Recall that a bundle map between $p|_Y$ and $q|_Y$ corresponds to a section in $\Gamma(\mathrm{Hom}(p,q), Y)$. Let s be a such a section that corresponds to an isomorphism between $p|_Y$ and $q|_Y$. By Lemma 1.29.2, there is a section $\hat{s} \in \Gamma(X, \mathrm{Hom}(p,q))$ such that $\hat{s}(y) = s(y)$ for any $y \in Y$. Define:

$$U = \{x \in X \mid \hat{s}(x) \in \mathrm{Hom}(p,q)^{-1}(x) = \mathrm{hom}(p^{-1}(x), q^{-1}(y)) \text{ is an iso}\}$$

1.29.4 Excercise. Show that the set U is open in X.

By definition, the section $\hat{s} \in \Gamma(U, \mathrm{Hom}(p,q))$ gives an isomorphism between $p|_U$ and $q|_U$. □

We can now prove the theorem.

Proof of Theorem 1.29.1. To prove the theorem we need to show that for any complex vector bundle $p : E \longrightarrow X$, the vector bundles $f^*p : F_0 \longrightarrow Y$ and $g^*p : F_1 \longrightarrow Y$ are isomorphic. Let $H : Y \times I \longrightarrow X$ be a homotopy between$f$ and g, i.e., H is a continuous function such that $H(y,0) = f(y)$ and $H(y,1) = g(y)$. For $t \in I$, let $h_t : Y \longrightarrow X$ be the map given by the formula $h_t(y) = H(y,t)$. We are going to study the following function of sets which we denote by $\alpha : I \longrightarrow \mathrm{Vect}(Y)$:

$$I \ni t \mapsto \alpha(t) := h_t^*p \in \mathrm{Vect}(Y)$$

We think about $\mathrm{Vect}(Y)$ as a discreet topological space and we claim that the above function is continuous. Note that if this is the case then, since I is path connected, the image of this map has to be one point in the discreet space $\mathrm{Vect}(Y)$. This would shows that $f^* = h_0^*p$ and $g^* = h_1^*p$ are isomorphic and prove the theorem.

To prove our claim we need to show that $\alpha^{-1}(q)$ is an open subset of I. If this set is empty then it is open. Assume that $\alpha^{-1}(q)$ is not empty and let $t \in I$ be such that $\alpha(t) = q$, i.e., q is isomorphic to h_t^*p. We need to show that there is some $\epsilon > 0$ for which the open interval $(t - \epsilon, t + \epsilon) \subset \alpha^{-1}(q)$, i.e, for any $s \in (t - \epsilon, t + \epsilon)$, h_s^*p is isomorphic to h_t^*p.

Consider the following two maps:

$$H : Y \times I \longrightarrow X \qquad Y \times I \xrightarrow{\mathrm{pr}} Y \xrightarrow{h_t} X, \quad G = h_t \circ \mathrm{pr}$$

and the pull-back vector bundles H^*p and G^*p. Note that the vector bundles $H^*p|_{Y\times\{t\}}$ and $G^*p|_{Y\times\{t\}}$ are isomorphic to h_t^*p. It follows from Lemma 1.29.3, that there is an open subset $Y\times\{t\}\subset U\subset Y\times I$ such that $H^*p|_U$ and $G^*p|_U$ are isomorphic. Since Y is compact there is $\epsilon>0$ for which $Y\times(t-\epsilon,t+\epsilon)\subset U$. It then follows that the following restricted bundles are also isomorphic:

$$H^*p|_{Y\times(t-\epsilon,t+\epsilon)} \qquad G^*p|_{Y\times(t-\epsilon,t+\epsilon)}$$

It follows that for any $s\in(t-\epsilon,t+\epsilon)$ the bundles $H^*p|_{Y\times\{s\}}$ and $G^*p|_{Y\times\{s\}}$ are isomorphic too. Notice that $H^*p|_{Y\times\{s\}}$ is h_s^*p and $G^*p|_{Y\times\{s\}}$ is h_t^*p. We can conclude that for any $s\in(t-\epsilon,t+\epsilon)$, the bundles h_s^*p and h_t^*p are isomorphic, which is what we aimed to prove. □

Theorem 1.29.1 has a lot of important consequences:

1.29.5 Corollary. *(1) If $f:Y\longrightarrow X$ is a homotopy equivalence between compact spaces, then* $\mathrm{Vect}(f):\mathrm{Vect}(X)\longrightarrow\mathrm{Vect}(Y)$ *is a bijection.*

(2) If X is a contractible and compact space, then $\mathrm{Vect}(X)$ *is isomorphic to* $\mathbf{N}$ *and any vector bundle over X is isomorphic to the product bundle* $\mathbf{n}$ *for some $n\in\mathbf{N}$.*

(3) Any vector bundle over a non-empty, compact, and convex subset in $\mathbf{R}^l$ *is isomorphic to the product bundle* $\mathbf{n}$ *for some $n\in\mathbf{N}$.*

Proof. (1): Let $g:X\longrightarrow Y$ be such that gf is homotopic to id_Y and fg is homotopic to id_X. Recall that $\mathrm{Vect}(gf)=\mathrm{Vect}(f)\mathrm{Vect}(g)$ and $\mathrm{Vect}(fg)=\mathrm{Vect}(g)\mathrm{Vect}(f)$. Thus according to 1.29.1, we have:

$$\mathrm{id}=\mathrm{Vect}(\mathrm{id})=\mathrm{Vect}(gf)=\mathrm{Vect}(f)\mathrm{Vect}(g)$$

$$\mathrm{id}=\mathrm{Vect}(\mathrm{id})=\mathrm{Vect}(fg)=\mathrm{Vect}(g)\mathrm{Vect}(f)$$

The functions $\mathrm{Vect}(f)$ and $\mathrm{Vect}(g)$ are therefore bijections.

(2): If X is contractible, then the map $X\longrightarrow D^0$ is a homotopy equivalence. It follows from statement (1) that the induced function $\mathbf{N}=\mathrm{Vect}(D^0)\longrightarrow\mathrm{Vect}(X)$ is a bijection.

(3): This is a consequence of statement (2) since any non-empty convex subset of $\mathbf{R}^l$ is contractible □

The above statements about vector bundle imply directly the corresponding statements for K-theory:

1.29.6 Corollary. *(1) Let Y be a compact space and $f,g:Y\longrightarrow X$ be continuous maps. If f and g are homotopic, then $K(f)=K(g)$.*

(2) If $f : Y \longrightarrow X$ is a homotopy equivalence between compact spaces, then $K(f) : K(X) \longrightarrow K(Y)$ is an isomorphism of commutative rings.

(3) If X is a is a contractible and compact space, then $K(X)$ is isomorphic to $\mathbf{Z}$.

(4) If X is a non-empty, compact, and convex subset in $\mathbf{R}^l$, then $K(X) = \mathbf{Z}$.

1.30 Rank of a vector bundle

Let $p : E \longrightarrow X$ be a complex vector bundle. A point $x \in X$ can be identified with a map denoted by the same symbol $x : D^0 \longrightarrow X$. Using this map we can define the function $\operatorname{rank}_x : \operatorname{Vect}(X) \longrightarrow \operatorname{Vect}(D^0) = \mathbf{N}$ by the formula:

$$\operatorname{rank}_x(p) := x^*p = \dim(p^{-1}(x))$$

Extend this definition to the function denoted by the same symbol $\operatorname{rank}_x : K(X) \longrightarrow K(D^0) = \mathbf{Z}$.

The rank is constant on the path-components of X:

1.30.1 Proposition. *Let $p : E \longrightarrow X$ be a complex vector bundle. If x and y belong to the same path component of X, then:*

$$\operatorname{rank}_x(p) = \operatorname{rank}_y(p)$$

Proof. If x and y belong to the same path component of X, then the maps $x, y : D^0 \longrightarrow X$ are homotopic. The proposition then follows from 1.29.1. □