Lecture 7

1.31 Clutching functions and homotopical properties II

We define $\mathrm{Vect}_k(X)$ to be the subset of $\mathrm{Vect}(X)$ consisting of the equivalence classes of vector bundles whose rank is k.

Consider a map $\pi : D^n_- \coprod D^n_+ \longrightarrow S^n$ given by the formula:

$$\pi(x) = \begin{cases} (x, \sqrt{1-|x|^2}) & \text{if } x \in D^n_+ \\ (x, -\sqrt{1-|x|^2}) & \text{if } x \in D^n_- \end{cases}$$

Note that S^n has the quotient topology induced by this map. In this way S^n is isomorphic to the quotient of $D^n_- \coprod D^n_+$ by the equivalence relation that identifies $x \in S^{n-1} \subset D^n_-$ with the same point $x \in S^{n-1} \subset D^n_+$. Let us fix a map $f : S^{n-1} \longrightarrow GL(\mathbf{C}^k)$. Define E_f to be the quotient of the space:

$$(D^n_- \times \mathbf{C}^k) \coprod (D^n_+ \times \mathbf{C}^k)$$

by the equivalence relation that identifies $(x, v) \in S^{n-1} \times \mathbf{C}^k \subset D^n_- \times \mathbf{C}^k$ with $(x, f(x)(v)) \in S^{n-1} \times \mathbf{C}^k \subset D^n_+ \times \mathbf{C}^k$. Let $\pi_f : E_f \longrightarrow S^n$ be the map which is given by the projection onto the first component. Note that $\pi_f^{-1} = \mathbf{C}^k$ is a complex vector space.

1.31.1 Proposition. *$\pi_f : E_f \longrightarrow S^n$ is a complex vector bundle.*

Consider a map $\pi : D^n_- \times I \coprod D^n_+ \times I \longrightarrow S^n \times I$ given by the formula:

$$\pi(x,t) = \begin{cases} ((x, \sqrt{1-|x|^2}), t) & \text{if } x \in D^n_+ \\ ((x, -\sqrt{1-|x|^2}), t) & \text{if } x \in D^n_- \end{cases}$$

Note that $S^n \times I$ has the quotient topology induced by π. In this way $S^n \times I$ is isomorphic to the quotient of $D^n_- \times I \coprod D^n_+ \times I$ by the equivalence relation that identifies $(x, t) \in S^{n-1} \times I \subset D^n_- \times I$ with the same point $(x, t) \in S^{n-1} \times I \subset D^n_+ \times I$

Let us fix a map $H : S^{n-1} \times I \longrightarrow GL(\mathbf{C}^k)$. Define E_H to be the quotient of the space:

$$(D^n_- \times I \times \mathbf{C}^k) \coprod (D^n_+ \times I \times \mathbf{C}^k)$$

by the equivalence relation that identifies:

$$(x, t, v) \in S^{n-1} \times I \times \mathbf{C}^k \subset D^n_- \times I \times \mathbf{C}^k$$

with

$$(x, t, H(x,t)) \in S^{n-1} \times I \times \mathbf{C}^k \subset D^n_+ \times I \times \mathbf{C}^k$$

Let $\pi_H : E_H \longrightarrow S^n \times I$ be the map which is given by the projection onto the first two components. Note that $\pi_f^{-1} = \mathbf{C}^k$ is a complex vector space.

1.31.2 Proposition. *$\pi_H : E_H \longrightarrow S^n \times I$ is a complex vector bundle.*

Let $i_0, i_1 : S^n \subset S^n \times I$ be the inclusions that map x to respectively $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$. Note that the pullback $i_0^*\pi_H$ is a vector bundle isomorphic to $\pi_{H(-,0)}$. Analogously the pullback $i_1^*\pi_H$ is a vector bundle isomorphic to $\pi_{H(-,1)}$.

1.31.3 Proposition. *Assume that maps $f, g : S^{n-1} \longrightarrow GL(\mathbf{C}^k)$ are homotopic. Then the vector bundles $\pi_f : E_f \longrightarrow S^n$ and $\pi_g : E_g \longrightarrow S^n$ are isomorphic.*

Proof. Let $H : S^{n-1} \times I \longrightarrow GL(\mathbf{C}^k)$ be a homotopy between f and g. Consider the induced vector bundle $\pi_H : E_H \longrightarrow S^n \times I$. Since $S^n \times I$ is compact and the inclusions $i_0, i_1 : S^n \longrightarrow S^n \times I$ are homotopic, according to Corollary 1.29.6, the pull-backs $i_0^*\pi_H$ and $i_1^*\pi_H$ are isomorphic. As these pull-backs can be identified with $\pi_f : E_f \longrightarrow S^n$ and $\pi_g : E_g \longrightarrow S^n$, the proposition is proven. □

According to the above proposition we have a well defined function:

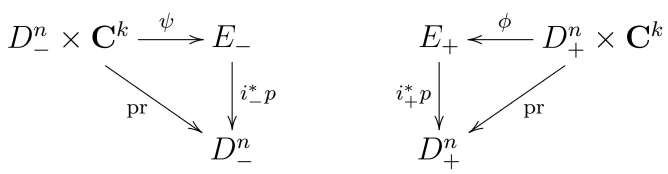
$$\mathcal{F}_k : [S^{n-1}, GL(\mathbf{C}^k)] \longrightarrow \mathrm{Vect}_k(S^n), \quad \mathcal{F}_k([f]) := (\pi_f : E_f \longrightarrow S^n)$$

1.31.4 Theorem. *The function $\mathcal{F}_k : [S^{n-1}, GL(\mathbf{C}^k)] \longrightarrow \mathrm{Vect}_k(S^n)$ is a bijection.*

To prove the theorem we are going to construct the inverse of $\mathcal{F}_k$. Let $p : E \longrightarrow S^n$ be a complex vector bundle. Let $i_- : D^n_- \longrightarrow S^n$ and $i_+ : D^n_+ \longrightarrow S^n$ be maps defined by the formulas:

$$i_-(x) := (x, -\sqrt{1 - |x|^2}), \qquad i_+(x) := (x, \sqrt{1 - |x|^2})$$

Since D^n is contractible, the pullbacks $i_-^*p : E_- \longrightarrow D^n_-$ and $i_+^*p : E_+ \longrightarrow D^n_+$ are the product bundles. Let us choose isomorphisms:



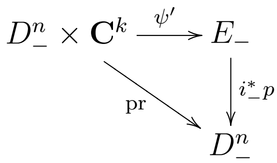
Define $\mathcal{G}_k(p) : S^{n-1} \longrightarrow GL(\mathbf{C}^k)$ to be the unique function for which the following equality holds:

$$\phi^{-1}\psi(x, v) = (x, \mathcal{G}_k(p)(v))$$

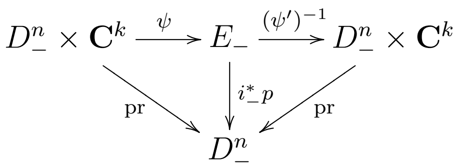
Explicitly

$$\mathcal{G}_k(p)(x)(v) = \mathrm{pr}_{\mathbf{C}^k}\phi^{-1}\psi(x, v)$$

We claim that the homotopy type of this map $\mathcal{G}_k(p) : S^{n-1} \longrightarrow GL(\mathbf{C}^k)$ is independent of the choices of the trivializations ψ and ϕ. Assume that instead of ψ, we chose another trivialization:



Consider the composition:



Let $g : D^n_- \longrightarrow GL(\mathbf{C}^k)$ be the map induced by the isomorphism $\psi'^{-1}\psi$. Since $GL(\mathbf{C}^k)$ is path connected and D^n_- is contractible, the map $g : D^n_- \longrightarrow GL(\mathbf{C}^k)$ is homotopic to the constant map $\mathrm{id} : D^n_- \longrightarrow GL(\mathbf{C}^k)$ with value the identity element in $GL(\mathbf{C}^k)$. It follows that $g\mathcal{G}_k(p)$ and $\mathcal{G}_k(p)$ are homotopic. Note however that $g\mathcal{G}_k(p)$ is the corresponding map with respect to the trivializations ψ' and ϕ of p. Same argument can be used to show independence of the homotopy type of $\mathcal{G}_k(p)$ with respect to the trivialization ϕ.

We can conclude that we have a well define function:

$$\mathcal{G}_k : \mathrm{Vect}_k(S^n) \longrightarrow [S^{n-1}, GL(\mathbf{C}^k)]$$

From the definitions of $\mathcal{F}_k$ and $\mathcal{G}_k$ it is clear that they are inverse isomorphisms proving Theorem 1.31.4

1.32 The ring structure

The sets $\mathrm{Vect}_k(S^n)$, for $k \geq 0$, come with:

- elements $\mathbf{k} \in \mathrm{Vect}_k(S^n)$, which correspond to product bundles $\mathbf{k} : S^n \times \mathbf{C}^k \longrightarrow S^n$, for $k \geq 0$;

- addition $\mathrm{Vect}_k(S^n) \times \mathrm{Vect}_l(S^n) \ni (p, q) \mapsto p \oplus q \in \mathrm{Vect}_{k+l}(S^n)$;

- multiplication$\mathrm{Vect}_k(S^n) \times \mathrm{Vect}_l(S^n) \ni (p, q) \mapsto p \otimes q \in \mathrm{Vect}_{kl}(S^n)$;

Analogously, the sets $[S^{n-1}, GL(\mathbf{C}^k)]$, for $k \geq 0$, come with:

- elements $[\iota_k : S^{n-1} \longrightarrow GL(\mathbf{C}^k)]$, which correspond to the constant map whose value is the identity element in $GL(\mathbf{C}^k)$;

- addition:

$$[S^{n-1}, GL(\mathbf{C}^k)] \times [S^{n-1}, GL(\mathbf{C}^l)] \ni (f, g) \mapsto f \oplus g \in [S^{n-1}, GL(\mathbf{C}^{k+l})]$$

- multiplication

$$[S^{n-1}, GL(\mathbf{C}^k)] \times [S^{n-1}, GL(\mathbf{C}^l)] \ni (f, g) \mapsto f \otimes g \in [S^{n-1}, GL(\mathbf{C}^{kl})]$$

By direct verification one can show:

1.32.1 Proposition. *The functions $\mathcal{F}_k : [S^{n-1}, GL(\mathbf{C}^k)] \longrightarrow \mathrm{Vect}_k(S^n)$, for $k \geq 0$, satisfies the following properties:*

(1) $\mathcal{F}_k(\iota_k) = \mathbf{k}$.

(2) Let $f \in [S^{n-1}, GL(\mathbf{C}^k)]$ and $g \in [S^{n-1}, GL(\mathbf{C}^l)]$. Then $\mathcal{F}_{k+l}(f \oplus g) = \mathcal{F}_k(f) \oplus \mathcal{F}_l(g)$.

(3) Let $f \in [S^{n-1}, GL(\mathbf{C}^k)]$ and $g \in [S^{n-1}, GL(\mathbf{C}^l)]$. Then $\mathcal{F}_{kl}(f \otimes g) = \mathcal{F}_k(f) \otimes \mathcal{F}_l(g)$.

1.32.2 Excercise. Proof the above proposition.