Lecture 8

1.33 Tautological line bundle over S^2

Consider the tautological line bundle $\lambda : E \longrightarrow \mathbf{C}P^1 = S^2$. In this section we are going to identify the clutching function $\mathcal{G}_1(\lambda) \in [S^1, GL(\mathbf{C})]$ and study its properties. Note that the group $GL(\mathbf{C})$ can be identified with the multiplicative group of non-zero complex numbers $\mathbf{C}^*$. Let $D_- \subset \mathbf{C}P^1$ be the subspace of lines in $\mathbf{C}^2$ that are generated by vectors of the form $(z, 1)$ for $|z| \leq 1$. The $\mathbf{C}$-line containing vector $(z, 1)$ is denoted by $L(z, 1)$. Let $D_+ \subset \mathbf{C}P^1$ be the subspace of lines in $\mathbf{C}^2$ that are generated by vectors of the form $(1, z)$ for $|z| \leq 1$. The $\mathbf{C}$-line containing vector $(1, z)$ is denoted by $L(1, z)$. Note that the functions:

$$\mathbf{C} \supset D^2 = \{z \mid |z| \leq 1\} \ni z \mapsto L(z, 1) \in D_-$$

$$\mathbf{C} \supset D^2 = \{z \mid |z| \leq 1\} \ni z \mapsto L(1, z) \in D_+$$

are isomorphisms. Thus we can think about D_- and D_+ as discs. Note further that $D_- \cap D_+ = \{z \in \mathbf{C} \mid |z| = 1\}$ is the circle $S^1 \subset \mathbf{C}$.

Let $\phi : D^2_- \times \mathbf{C} \longrightarrow \lambda^{-1}(D_-)$ and $\psi : D^2_+ \times \mathbf{C} \longrightarrow \lambda^{-1}(D_+)$ be maps defined by the formulas:

$$\phi(L(z, 1), t) := (L(z, 1), t(z, 1)) \qquad \psi(L(1, z), t) := (L(1, z), t(1, z))$$

These maps are trivializations of λ over the subspaces D^2_- and D^2_+. Consider the composition:

$$S^1 \times \mathbf{C} = D^2_- \cap D^2_+ \times \mathbf{C} \xrightarrow{\psi^{-1}\phi} D^2_- \cap D^2_+ \times \mathbf{C} = S^1 \times \mathbf{C}$$

It maps an element $(L(z, 1), t)$ to $(L(1, 1/z), tz)$. Thus the induced map $S^1 \longrightarrow GL(\mathbf{C})$ is the standard inclusion $S^1 \subset \mathbf{C}^* = GL(\mathbf{C})$. This proves:

1.33.1 Proposition. *The element $\mathcal{G}_1(\lambda) \in [S^1, GL(\mathbf{C})]$, associated with the tautological bundle $\lambda : E \longrightarrow \mathbf{C}P^1 = S^2$, is represented by the standard inclusion $S^1 \subset \mathbf{C}^* = GL(\mathbf{C})$.*

Consider the following vector bundles of rank 2 over $\mathbf{C}P^1 = S^2$:

$$(\lambda \otimes \lambda) \oplus \mathbf{1} \qquad \lambda \oplus \lambda$$

It follows from the above proposition, that the elements:

$$\mathcal{G}_2((\lambda \otimes \lambda) \oplus \mathbf{1}) \in [S^1, GL(\mathbf{C}^2)] \ni \mathcal{G}_2(\lambda \oplus \lambda)$$

are represented by the following functions:

$$S^1 \ni z \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \in GL(\mathbf{C}^2)$$

$$S^1 \ni z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in GL(\mathbf{C}^2)$$

The space $GL(\mathbf{C}^2)$ is connected. Thus there is a path $\omega : I \longrightarrow GL(\mathbf{C}^2)$ such that:

$$\omega(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \omega(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that since $\det(\omega(0)) = 1$ and $\det(\omega(1)) = -1$, these matrices belong to different connected components of $GL(\mathbf{R}^2)$. Thus the fact that we use complex numbers is **important**. We can use this path to define a homotopy:

$$H : S^1 \times I \longrightarrow GL(\mathbf{C}^2),\ H(z,t) := \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \omega(t) \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \omega(t)$$

By direct verification one can check that:

$$H(z,0) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \qquad H(z,1) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$$

We can thus conclude that:

1.33.2 Proposition. *Let $\lambda : E \longrightarrow \mathbf{C}P^1 = S^2$ be the tautological bundle. Then the bundles $(\lambda \otimes \lambda) \oplus \mathbf{1}$ and $\lambda \oplus \lambda$ are isomorphic.*

1.34 Statement of the periodicity theorem

Consider the map $p : S^2 \longrightarrow D^0$. It induces a homomorphism of rings $K(p) : K(D^0) \longrightarrow K(S^2)$. Recall that $K(D^0)$ is isomorphic to $\mathbf{Z}$ via the rank map. The homomorphism $K(p)$ assigns to $k \geq 0$ the element in $K(S^2)$ represented by the product bundle $\mathbf{k}$. Recall that $\lambda_1 : E \longrightarrow S^2 = \mathbf{C}P^1$ denotes the tautological line bundle (of rank 1). We can use $K(p)$ to define a ring homomorphism:

$$\mathbf{Z}[T] \ni (a_0 + a_1 T + \cdots + a_n T^n) \mapsto \mathbf{a_0} \oplus (\mathbf{a_0} \otimes \lambda_1) \oplus \cdots \oplus (\mathbf{a_0} \otimes \lambda_1^{\otimes n}) \in K(S^2)$$

which we denote by $\Lambda : \mathbf{Z}[T] \longrightarrow K(S^2)$. The polynomial $(T-1)^2 = T^2 - 2T + 1$ is sent via Λ to $(\lambda \otimes \lambda) \oplus \mathbf{1} - \lambda \oplus \lambda$, which is the zero element according to Proposition 1.33.2. Thus the homomorphism Λ factors through $\mathbf{Z}[T]/(T-1)^2 \longrightarrow K(S^2)$, which we also denote by Λ.

Further let X be a compact space. Consider the product $S^2 \times X$ and the following commutative diagram where the maps p_1 and p_2 are the appropriate projections:

$$\begin{array}{ccc} X \times S^2 & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ S^2 & \longrightarrow & D^0 \end{array}$$

By applying K-theory to the above diagram, we get a commutative diagram of rings:

$$\begin{array}{ccc} K(D^0) & \longrightarrow & K(X) \\ \downarrow & & \downarrow{\scriptstyle K(p_1)} \\ K(S^2) & \xrightarrow{K(p_2)} & K(X \times S^2) \end{array}$$

Since $K(D^0) = \mathbf{Z}$, we can use the commutativity of the above diagram to get a ring homomorphism $\Omega : K(X) \otimes_{\mathbf{Z}} K(S^2) \longrightarrow K(X \times S^2)$.

1.34.1 Theorem. *The following ring homomorphisms are isomorphisms:*

$$\Lambda : \mathbf{Z}[T]/(T-1)^2 \longrightarrow K(S^2) \qquad \Omega : K(X) \otimes_{\mathbf{Z}} K(S^2) \longrightarrow K(X \times S^2)$$

1.35 Reduced K-theory

Let X be a space. For any point $x \in X$, we have a rank ring homomorphism: $\text{rank}_x : K(X) \longrightarrow \mathbf{Z}$.

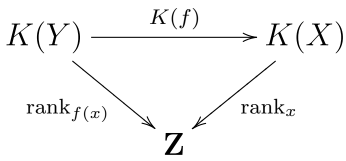
1.35.1 Definition. $\tilde{K}(X) := \{\xi \in K(X) \mid$ *for any* $x \in X, \text{rank}_x(\xi) = 0\}$.

Since rank_x is a ring homomorphism, $\tilde{K}(X)$ is an ideal in $K(X)$. Thus, if $\xi \in \tilde{K}(X)$, then for any $\tau \in K(X)$, $\xi \otimes \tau \in \tilde{K}(X)$. In particular for $\xi, \tau \in \tilde{K}(X)$, then $\xi \otimes \tau \in \tilde{K}(X)$. In this way $\tilde{K}(X)$ is a "commutative ring" without the identity.

Let $f : X \longrightarrow Y$ be a map. Consider the induced ring homomorphism $K(f) : K(Y) \longrightarrow K(X)$. For any point $x \in X$, since

$$\begin{array}{ccccc} & & D^0 & & \\ & {\scriptstyle x}\swarrow & & \searrow{\scriptstyle f(x)} & \\ X & & \xrightarrow{\quad f \quad} & & Y \end{array}$$

commutes, we get a commutative diagram of rings:



It follows that f induces a homomorphism of abelian groups $\tilde{K}(f) : \tilde{K}(Y) \longrightarrow \tilde{K}(X)$. It is actually a homomorphisms of "commutative rings" without identities.

Consider S^2. The rank homomorphism rank : $K(S^2) \longrightarrow \mathbf{Z}$, is given by the evaluation at 0 homomorphism ev : $\mathbf{Z}[T]/(T-1)^2 \longrightarrow \mathbf{Z}$. Its kernel $\tilde{K}(S^2)$, as an abelian group, is generated by $T-1$ and is isomorphic to $\mathbf{Z}$. Note that the multiplication $(T-1)(T-1) = 0$. So the multiplication on $\tilde{K}(S^2)$ is trivial.

1.36 Cones and exact sequences

Let X be a space and $A \subset X$ its subspace. Define X/A to be the following topological space. As a set, $X/A := (X \setminus A) \cup \{[A]\}$, i.e., in addition to elements of the complement $X \setminus A$ it has one more element which is denoted by $[A]$. Let $\pi : X \longrightarrow X/A$ be a function define by the formula:

$$\pi(x) = \begin{cases} x & \text{if } x \in X \setminus A \\ [A] & \text{if } x \in A \end{cases}$$

Define the topology on X/A to be the quotient topology given by the above function. Thus $U \subset X/A$ is open if and only if $\pi^{-1}(U)$ is open in X.

1.36.1 Excercise. Assume that X is compact and $A \subset X$ is closed. Show that X/A is compact.

Let X be a space. Consider the product $X \times I$ and a subspace $X_1 = X \times \{1\} \subset X \times I$. The space $(X \times I)/X_1$ is called the cone of X and denoted by CX. The inclusion $X = X \times \{0\} \subset (X \times I)/X_1 = CX$ is called the base of the cone.

1.36.2 Excercise. Show that CX is a contractible space for any X.

Let $f : A \longrightarrow X$ be a map. Define $X \cup_f CA$ to be the quotient of $X \coprod CA$ by the relation that identifies a in the base of CA with $f(a)$ with the quotient topology given by the quotient map $X \coprod CA \longrightarrow X \cup_f CA$. The composition of this map with the inclusion $X \subset X \coprod CA$, is denoted by $i : X \longrightarrow X \cup_f CA$.

1.36.3 Excercise. Show that if A and X are compact, then so is $X \cup_f CA$, for any $f : A \longrightarrow X$.

1.36.4 Theorem. *Assume that $f : A \longrightarrow X$ is a map between compact spaces. Then the following is an exact sequence of abelian groups:*

$$\tilde{K}(A) \xleftarrow{\tilde{K}(f)} \tilde{K}(X) \xleftarrow{\tilde{K}(i)} \tilde{K}(X \cup_f CA)$$

1.36.5 Excercise. Prove the above theorem.