SF2724: APPLIED TOPOLOGY, SPRING 2011
SECOND PART: DE RHAM COHOMOLOGY

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For this part of the course we follow the book "From calculus to cohomology: de Rham cohomology and characteristic classes" by Madsen and Tornehave. Here some comments on the lectures.

Lecture 9, March 31. *Chapter 1:* Short illustration of the basic idea of de Rham cohomology in terms of vector fields on open sets of $\mathbb{R}^2$ and $\mathbb{R}^3$. *Chapter 2:* The alternating algebra. All is important, in particular the relation with the determinant (Lemma 2.13) and how to find a basis (Theorem 2.15).

Lecture 10, April 7. *Chapter 3:* Definition of de Rham cohomology for open subsets of $\mathbb{R}^n$. Important details are: differential of a function in standard basis (Example 3.3), characterization of d (Theorem 3.7), d commutes with pullback ϕ^*, and characterization of the pullback (Theorem 3.12). Pullback of 1-forms and n-forms (Example 3.13). *Chapter 4:* Basic homological algebra, long exact sequence of cohomology groups from a short exact sequence of chain complexes.

Lecture 11, April 14. *Chapter 5:* The Mayer-Vietoris sequence is an important tool in making (de Rham) cohomology a computable topological invariant. Try to understand the connecting homomorpism as explicitly as possible, see Definition 4.5, Theorem 5.1 (for surjectivity of J), and Exercise 5.1. *Chapter 6:* Homotopy invariance. Find a chain homotopy by integration (Theorem 6.7). Explicit computations (p. 43-46).

Lecture 12, April 28. *Chapter 7:* Classical applications of algebraic topology in Theorem 7.1 (Brouwer's fixed point theorem) and Theorem 7.3 (existence of non-vanishing vector fields on spheres). Both results follow from the knowledge of $H^*(\mathbb{R}^n \setminus \{0\})$ together with homotopy invariance. The rest of the chapter is also interesting, containing for example the Jordan-Brouwer separation theorem (Thm. 7.10), invariance of domain (Cor. 7.13), dimension invariance (Cor. 7.14). *Chapter 8:* We will only need the very basic definitions of smooth manifolds, where the most important concepts are charts and smooth atlases (Def. 8.3). All other concepts such as smooth functions, maps, diffeomorphisms are then defined by composing with the charts in an appropriate way. Further important concepts (not mentioned in the lecture) are submanifolds and embeddings (Def 8.8, 8.10), and the fact that any smooth manifold can be realized as a submanifold of $\mathbb{R}^N$ for N large enough (Thm. 8.11). *Chapter 9:* For analysis on manifolds it is fundamental to have a good definition of tangent vectors (see p. 65-67): One definition uses directions of curves, the other defines a tangent direction as a directional derivative acting on functions. See also `http://en.wikipedia.org/wiki/Tangent_space`.

Date: May 23, 2011.

Lecture 13, May 5. *Chapter 9:* The tangent bundle has transition functions given by the Jacobians of the coordinate-change maps for the atlas, see top of p. 68 and `http://en.wikipedia.org/wiki/Tangent_bundle`. The transition functions are maps $U_1 \cap U_j \to GL(\mathbb{R}^n)$, by composing with the map $GL(\mathbb{R}^n) \ni f \mapsto \mathrm{Alt}^k(f) \in GL(\mathrm{Alt}^k(\mathbb{R}^n))$ (see p. 13) one gets the transition functions for the bundle of k-forms. The k-forms on a manifold are sections of this bundle (in the book k-forms are equivalently defined as alternating multilinear maps of tangent vectors satisfying a smoothness condition, Def. 9.5). To define integration of forms it is necessary to work with oriented manifolds, important is the fact that a choice of orientation is equivalent to a choice of positive atlas (see p. 72). Not every manifold is orientable, a good example are the real projective spaces $\mathbb{R}P^n$ of even dimension (see Ex. 9.18 for the the orientation form S^n, and Ex. 9.19 for $\mathbb{R}P^n$). The rest of Chapter 9 contains a lot of interesting material: Tubular neighbourhoods (Thm. 9.23, Cor. 9.28), Cohomology of S^n (Ex. 9.29), Mayer-Vietoris (Rem. 9.30), Cohomology of $\mathbb{R}P^n$ (Ex. 9.31). *Chapter 10:* The integral of an n-form over an oriented n-manifold is defined in Lem. 10.1 and Prop. 10.2, here it is crucial to have an oriented atlas for the integral to be well-defined. Stokes' theorem (Thm 10.8) relates the exterior differential d to the integral. Theorem 10.13 relates integration to cohomology and is quite important, you should read the proof over the following pages. Corollary 10.14 shows that top-dimensional cohomology is always 1-dimensional for compact connected orientable manifolds (and in particular the sphere has smallest possible cohomology for such manifolds).

Lecture 14, May 12. *Chapter 11:* From Corollary 10.14 we know that $H^n(M^n) = \mathbb{R}$ for a compact connected manifold M^n equipped with an orientation, this gives the definition of the degree of a smooth map as the induced map on $\mathbb{R}$. In Theorem 11.19 the degree $\deg(f)$ (which is a global invariant) is computed as a sum of local contributions $\mathrm{Ind}(f;q)$. This is conveniently proved using differential forms since we can choose the form to have support precisely near the regular value and thus intermediate between local and global. With slightly different interpretation Proposition 11.11 tells us that $\deg(f)$ is a (co)bordism invariant (for definitions see `http://www.map.him.uni-bonn.de/Bordism`). Divide the boundary of X into two parts N_+ and N_-, where N_- is equipped with the opposite orientation. Then $\deg(f_+) = \deg(f_-)$ where $f_\pm = F_{|N_\pm}$. In this chapter two applications of the degree are given, first the linking number of submanifolds, second the index of vector fields. Theorem 11.14 gives the linking number both as an integral and a combinatorial sum (the sum explained further in Remark 11.15). The total index of a vector field X is defined as a sum of local indices $\iota(X;p)$ at zeros p of the vector field (note that the local index is automatically zero if $X(p) \neq 0$.) By Proposition 11.11 the index is given by the degree of the vector field restricted to the boundary of the domain, see Thm 11.22, Cor 11.24. This is finally applied to identify the index of a vector field on a manifold M^n with the degree of the Gauss map of a tubular neighborhood of an embedding $M^n \subset \mathbb{R}^{n+k}$, Thm 11.27. The conclusion is that neither side of the identity depends on anything more than M.

Lecture 15, May 19. *Chapter 12:* In this chapter the Index of a vector field X on a manifold M is identfied with the Euler characteristic $\chi(M)$, this is the Poincare-Hopf Theorem 12.1. Since we already know that the index does not depend on X we choose the vector field to be a gradient-like vector field for a Morse function f on

M, this is explained and defined in 12.2-12.8. If M is equipped with a Riemannian metric then the gradient vector field of f is defined in Remark 12.10. The first step of the proof is to identify the local indices at critical points of the Morse function, this is done in Lemma 12.9 and gives the expression for the index in Theorem 12.11. The second step of the proof is to study the sub-level sets $M(a)$ of the Morse function. The very important Lemmas 12.12 and 12.13 describe how the topology of $M(a)$ changes as a passes through a critical value, some pictures can be found at `http://en.wikipedia.org/wiki/Morse_theory`. This gives a simple formula for the Euler characteristic in Proposition 12.14, it is slightly more complicated to formulate the effect on cohomology (see Exercise 12.6). Theorem 12.16 finally computes the Euler characteristic as a sum over the critical points of the Morse function. There are several other nice applications of Morse theory, see Corollary 12.17, Examples 12.19-20, Exercises 12.9-11.