KTH Teknikvetenskap

SF2729 GROUPS AND RINGS
HOMEWORK ASSIGNMENT II
RINGS

The following six homework problems can count as the first problem in the second section of the final exam. The solutions should be handed in not later than April 26, 2011. The computations and arguments should be easy to follow. Collaborations should be clearly stated.

Credits on homework	31-35	26-30	21-25	16-20	11-15	6-10	0-5
Credits on problem 1 of part II	6	5	4	3	2	1	0

Problem 1. Let R be a ring with unity. For $r \in R$, let $ev_r : R[x] \to R$ be the evaluation map at r. Show that R is a commutative ring if and only if ev_r is a ring homomorphism for every $r \in R$. **(4)**

Problem 2. For a ring R with unity, denote by R^* the group of invertible elements (units).

(1) Let R_1 and R_2 be two rings with unity. Show that $R_1^* \times R_2^* \cong (R_1 \times R_2)^*$ (isomorphism as groups). **(2)**

(2) Use the previous part to show that if $n, m \in \mathbb{Z}$ are relatively prime then

$$\phi(n)\phi(m) = \phi(nm),$$

where ϕ denotes the Euler function. **(2)**

Problem 3.

(1) Let R be a ring. Define operations on $\mathbb{Z} \times R$ so that $\mathbb{Z} \times R$ is a ring with unity. **(3)**

(2) Show that every ring R can be embedded[1] in the ring of endomorphisms of an abelian group. **(5)**

Problem 4. Let R be a commutative ring with unity, $a \in R$ and let $I \subseteq R$ be an ideal. Consider the following set:

$$\mathcal{I} = \{f(x) \in R[x] \text{ such that } f(a) \in I\}.$$

(1) Show that $\mathcal{I}$ is an ideal of $R[x]$. **(2)**

(2) Show that $\mathcal{I}$ is prime if and only if I is prime. **(3)**

[1] A ring R is embedded in a ring S if there is a ring homomorphism $R \to S$ which is injective.

(3) Identify $\mathcal{I}$ for $R = \mathbb{Z}, a = 1$ and $I = (5)$. **(2)**

Problem 5. Let p be a prime number. Identify $\mathbf{Z}_{p^r}^*$, the group of units of the ring $\mathbf{Z}_{p^r}$ (note that the cases $p = 2$ and $p > 2$ are different). **(7)**

Problem 6. This problem is closely related to Problem 1. Let $R = \{0, a, b, c\}$. We wish to define a ring structure on R as follows. As an additive group, $R \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ (with neutral element 0). Multiplicatively, we have $a^2 = ab = a$ and $b^2 = ba = b$.

(1) Compute the products with c by using the distributive properties and show that R indeed becomes a ring in this manner. **(2)**
(2) Show that $ev_c = ev_0$ and explain why this map is a ring homomorphism. **(1)**
(3) Show that $ev_a = ev_b$ and explain why this map is a ring homomorphism. **(1)**
(4) Which conclusion do you draw after comparing the results obtained here with those of Problem 1? **(1)**