DIFFERENTIAL GEOMETRY, FALL 2011, READING INSTRUCTIONS

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Here comments and reading instructions following the course book

- *Manifolds and Differential Geometry*, Jeffrey M. Lee.

Other recommended books are

- *Introduction to smooth manifolds*, John M. Lee,
- *An introduction to differentiable manifolds and Riemannian geometry*, William M. Boothby,
- *Differentialgeometrie*, Christian Bär.
- *Foundations of differentiable manifolds and Lie groups*, Frank W. Warner
- *Semi-Riemannian geometry with applications to relativity*, Barrett O'Neill

Lecture 1, September 5. The goal of the first lecture was to introduce differentiable manifolds and smooth maps between them. *Section 1.1* collects some preliminaries, read carefully and watch out for misprints. *Section 1.2* introduces topological manifolds which are locally euclidean topological spaces. Here the topological concepts of Hausdorff, second countable, and paracompact spaces are introduced, skip this if unfamiliar (the paracompactness property is crucially used for the construction of a partition of unity in Section 1.5). The fundamental definitions for differentiable manifolds come in *Section 1.3*: chart, atlas, change of coordinate maps, smooth structure. It is important to understand carefully the concepts of equivalent atlases, and how a maximal atlas is related to a smooth structure. *Section 1.4* introduces smooth maps, the definition involves the fundamental idea of defining a property to hold on a manifold if it holds when the objects are transported to euclidean space using charts (for example there is a similar characterization of the topology induced by an atlas, see discussion after Lemma 1.30). Special cases of the definition cover diffeomorphisms and smooth functions. In Definition 1.57 the partial derivatives $\partial f/\partial x^i$ of a function f is defined in a chart $(U, \mathtt{x})$, it is important to understand in detail how the chain rule gives the relation to partial derivatives in a second chart $(V, \mathtt{y})$. On pages 26-27 there is a very interesting discussion on equivalent and non-equivalent differentiable structures.

Further reading. Even though I did not have time to mention it in the lecture you must read the very important *Section 1.5*. Smooth cut-off functions are constructed, and you should read the proof carefully. They are used to restrict a smooth function on a manifold to a function with support in a chart, while keeping the smoothness property. In Definition 1.70 a partition of unity is described, note that paracompactness is used in the existence proof. With a partition of unity any function smooth function on a manifold can be written as a sum $f = f \cdot 1 = f \sum_\alpha \varphi_\alpha = \sum_\alpha f \varphi_\alpha$,

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where each term $f\varphi_\alpha$ has support in a chart and the terms can thus be studied further as smooth functions with compact support on $\mathbb{R}^n$.

Recommended exercises. To help understand the material I suggest that you work through the following examples and exercises in the book: 1.38, 1.39, 1.40, 1.41, 1.42, 1.56, 1.61. (These are not homework exercises to be handed in, and of course there are too many to solve all of them in full detail. For the preliminary version of the book the corresponding examples and exercises are: 1.29, 1.30, 1.31, 1.32, 1.35, 1.53, 1.50.)

Lecture 2, September 12. In *Section 1.7* regular submanifolds are defined as an answer to the question "when is a subset of a manifold also a (sub)manifold?". As with most definitions on a smooth manifold the property is translated into a property on $\mathbb{R}^n$ using charts: in a chart a submanifold is a subset of an affine subspace of $\mathbb{R}^n$. In *Section 2.1* different definitions of the tangent vector space at a point p in a smooth manifold M are given. The first as equivalence classes of smooth curves through p. The second as vectors in $\mathbb{R}^n$ associated with a chart around p, together with a transformation property under changes of chart. The third as derivations (at p) acting on smooth functions on the manifold. The three definitions each have advantages and disadvantages, but *Section 2.2* gives isomorphisms showing that they all give the same tangent vector space T_pM. Think through these isomorphisms carefully. (Furthermore, in this section the cotangent space is introduced, we will come back to this in the next lecture.) In *Section 2.3* the tangent linear map $T_pf : T_pM \to T_{f(p)}N$ of a smooth map $f : M \to N$ is defined (other notation for T_pf such as df_p, Df_p, $(f_*)_p$, $f'(p)$ is also used). The definition of T_pf depends on which definition of tangent vectors is used, you must check that you understand each of the definitions and how to prove the chain rule $T_p(g \circ f) = T_{f(p)}g \circ T_pf$ in each case. Note a misprint in the definition Tangent map III: the defined derivation is an element of T_qN! It is interesting to note (at least for the first and third definitions) how simple the definitions of T_pf are and how easy it is to check the chain rule, the complications are hidden in the somewhat involved definitions of tangent vectors. This section also contains a version of the important inverse mapping theorem, Theorem 2.25.

Further reading. You should have a look at *Section 1.8* where manifolds with boundary are introduced, the definition is a straight-forward generalization of the definition of a smooth manifold (without boundary). Manifolds with boundary will be important in particular when we later see how to integrate differential forms and find the general Stokes's Theorem. You should also look through *Section 2.4* which contains a careful identification of the tangent space of a product manifold.

Recommended exercises. As mentioned above: (i) Work out in detail the isomorphisms between the three definitions of the tangent space, (ii) check that the three definitions of the tangent map are equivalent, (iii) check the chain rule using the three definitions, (iv) write out in coordinates the tangent map in charts (second definition).

Lecture 3, September 19. In this lecture we covered three subjects. First the definition of cotangent vectors and the cotangent space in *Section 2.2*. Although T_pM and T_p^*M have the same dimension and therefore are isomorphic as vector spaces there is no “natural” choice of isomorphism, only using some further information such as the choice of a chart or a Riemannian metric can we define an isomorphism $T_pM \to T_p^*M$. With a chart around p given, a basis $\{dx^i\}$ of T_p^*M is defined as the dual basis to the basis $\{\partial/\partial x^i\}$ of T_pM. The differential $df(p)$ at p of a function $f : M \to \mathbb{R}$ is introduced in *Section 2.3*. It is a cotangent vector and the definition is a slight simplification of the definition of the tangent map T_pf. Note the discussion on the middle of page 69: the differentials dx^i of the coordinate functions do have the same property as the dual basis. The second theme of the lecture was critical points and values, and the level submanifold theorem. In *Section 2.5* critical points and values are introduced, and Sard's theorem is stated. The important Theorem 2.4.1 in *Section 2.6* gives a way of checking that the level sets of a smooth map are regular submanifolds. Look at examples 2.43 and 2.44. The third subject of the lecture was the tangent bundle in *Section 2.7*, read carefully the construction of charts and the proof of Proposition 2.55. The parallel construction for the cotangent bundle is on page 85-86. Vector fields are sections of the tangent bundle and are introduced in *Section 2.8* (note that the differential df of a function is a section of the cotangent bundle). A vector field on M can be interpreted as a global derivation on $C^\infty(M)$ (Def. 2.69), the Lie derivative of a function with respect to a vector field is introduced, $\mathcal{L}_X f = Xf = df(X)$, (Def. 2.70).

Further reading. Have a look at *Section 2.5.1* where the Morse Lemma is stated and proved. This important result is fundamental in differential topology, see `http://en.wikipedia.org/wiki/Morse_theory`. Also, read the warning on page 121-122: the basis vector $\partial/\partial x^1$ does not only depend on the coordinate function x^1 but on the entire coordinate system $(x^1, \dots, x^n)$.

Recommended exercises. Fill in the details of Example 2.44. Exercises 2.54, 2.56.

Lecture 4, September 26. Todays lecture was about vector fields, *Section 2.8* in the book. A vector field is a smooth map $X : M \to TM$ such that $X_p \in T_pM$ for all $p \in M$. Theorem 2.72 gives an alternative characterization of vector fields as (global) derivations on $C^\infty(M)$. From Theorem 2.73 and Corollary 2.74 it then follows that the commutator (or Lie bracket) $[X, Y]$ of vector fields X, Y is again a vector field. A priori the composition of two vector fields should be a second order derivation acting on functions, but the commutator turns out to be of first order and thus a tangent vector. The Lie bracket is bilinear, skew-symmetric, and satisfies the Jacobi identity, see Proposition 2.76. This makes the vector space of vector fields on M into a Lie algebra. *Subsection 2.8.1* treats integral curves and flows. An integral curve of a vector field is a curve whose velocity vector at each point coincides with the vector field at the point. Just as tangent vectors naturally appear as derivatives of curves, vector fields appear as derivatives of flows. The concept of a (complete) flow and its associated vector field is on page 95. The reverse construction of a (maximal) flow for a given vector field is on pages 95-101. A vector field is complete if its flow (and thus all its integral curves) can be defined for all time. Note the important Lemma 2.99: a vector field with compact support (for example any vector field on a compact manifold) is complete. The Lie derivative is introduced in *Subsection 2.8.2*. It has a characterization in terms of flows: the Lie derivative $\mathcal{L}_X Y$ of Y with respect to X is the derivative of Y along the flow of X, read the proof of Proposition 2.105 carefully. Note that the definition of $\mathcal{L}_X Y$ requires both X and Y to be vector fields, it is not sufficient to know the value X_p

of X at a point to compute the Lie derivative with respect to X. Therefore this is not a satisfactory construction of "directional derivative" of vector fields, and in fact no such thing can be constructed without the introduction of some additional structure on the manifold. Theorem 2.109 states that vector fields commute if and only if their flows commute, Theorem 2.112 then gives a further description of the Lie bracket as a measure of how much the flows fail to commute. Finally, Theorem 2.113 tells us that commuting vector fields which are also linearly independent are coordinate vector fields for some chart.

Further reading. In *Section 2.9* covector fields or 1-forms are introduced. We will return to this later in connection with Chapter 8, but you should read this section now. 1-forms are the correct objects to integrate along curves, see *Section 2.10* (this should all be familiar from multi-variable calculus!) *Section 2.11* contains a short discussion on moving frames or frame fields.

Recommended exercises. (i) Lie Bracket in coordinates, Exercise 2.77. (ii) Completeness of vector fields, Exercise 2.90. (iii) Prove Theorem 2.112. (iv) Problem 13, page 123 (this defines the Hessian, or the second derivatives, of a function at a point where the first derivatives already vanish. In general second derivatives cannot be defined without further structure on the manifold.) (v) Show that the vector fields $X_1 = x\partial/\partial y - y\partial/\partial x$ and $X_2 = x\partial/\partial x + y\partial/\partial y$ on $\mathbb{R}^2$ commute, then find local coordinates for which they are the coordinate fields.

Lecture 5, October 3. This lecture was about immersions and submanifolds, *Sections 3.1-3.2* in the book. An immersion $f : M \to N$ is a smooth map such that $T_pf : T_pM \to T_{f(p)}N$ is injective for each $p \in M$. Further, f is said to be an embedding if $f : M \to F(M)$ is a homeomorphism. The image $f(M)$ of an embedding is a regular submanifold as defined in Section 1.7. After embeddings also injective immersions are important. If M is compact then an injective immersion is automatically an embedding (Exercise 3.6). Definition 3.13 defines $S \subset M$ to be an immersed submanifold if S is a smooth manifold and the inclusion map $S \to M$ is an injective immersion. In the book two further types of immersions are introduced: proper and weak embeddings. The corresponding submanifolds are called proper and weakly embedded. These are less important than regular and immersed submanifolds, but reflects the fact that several different definitions of submanifolds are used in the literature. The (weak) Whitney imbedding theorem, Theorem 3.21, states that every compact manifold M of dimension n can be embedded in $\mathbb{R}^{2n+1}$. The proof uses Sard's theorem which for maps $f : N_1 \to N_2$ tells us that the image $f(N_1)$ has measure zero if $\dim N_1 < \dim N_2$ since then all points in N_1 are critical points. Note that the induction part of the proof can be extended one step further for the immersion property (but not for injectivity) to conclude that M can be immersed into $\mathbb{R}^{2n}$. Specifically, we can find z outside the image of the map $df : TM \to \mathbb{R}^d$ also if $d > 2n$. The strong Whitney embedding and immersion theorems improves the above by one dimension and states that M can be embedded in $\mathbb{R}^{2n}$ and immersed in $\mathbb{R}^{2n-1}$.

Further reading. Read *Section 3.3* on submersions.

Recommended exercises. Exercises 3.6-3.9, 3.12 on page 129 (some of these are mainly exercises in topology, skip if unfamiliar!), Problem (2) on page 140.

Lecture 6, October 10. In Chapter 4 geometric concepts on smooth manifolds are introduced in two simple special cases, curves and hypersurfaces in Euclidean space. We will come back to these constructions in general in Chapter 13.

From *Section 4.1* we need only learn the definition of parametrized curves, their length, curvature etc., you may skip the material after Definition 4.4. *Section 4.2* starts with the definition of the standard covariant derivative $\overline{\nabla}$ of vector fields on $\mathbb{R}^n$. The construction relies on the fact that there is a constant frame field $\hat{e}_1, \dots, \hat{e}_n$ defined on all of $\mathbb{R}^n$, so that the derivative of a vector field is computed by differentiating its coefficients. This construction is not possible in general on a smooth manifold. Near a point there are always two unit normal fields on a hypersurface $M \subset \mathbb{R}^n$, the hypersurface is orientable if they can be extended to all of M. For hypersurfaces in Euclidean space the curvature is completely encoded in the bending of the unit normal field N. This bending is computed by the shape operator $S_N(V_p) := -\overline{\nabla}_{V_p} N$, which is a linear map from T_pM to itself. To compute the derivative here we extend the vector field N to an open neighborhood of p in $\mathbb{R}^n$, see the last lines on page 154. The shape operator is self-adjoint with respect to the induced Riemannian metric on M, read the proof of Proposition 4.19. Proposition 4.25 relates the curvature of a curve in M to the shape operator of M, the normal part of its acceleration vector is determined by the shape operator of M, see the discussion at the end of page 158. If the curve has only got a normal component of its acceleration then it is a geodesic, see definition 4.27. You may skip the rest of the section after this definition.

Further reading. Have a look at the rest of *Section 4.1*, pages 147-152, and the rest of *Section 4.2*, pages 160-165.

Recommended exercises. Problem (13), page 188.

Lecture 7, October 17. This lecture covered *Section 4.3*. The Levi-Civita covariant derivative on a hypersurface $M \subset \mathbb{R}^n$ is defined by $\nabla_{X_p} Y := \mathrm{proj}_{T_pM} \overline{\nabla}_{X_p} Y$ where $X_p \in T_pM$, Y is a vector field, and proj_{T_pM} denotes orthogonal projection onto T_pM. The Gauss formula (4.2) relates ∇, $\overline{\nabla}$ and the shape operator S_N. The Levi-Civita covariant derivative is torsion free and metric, and it is uniquely defined by these properties (see Proposition 4.43). In a chart ∇ is specified by the Christoffel Symbols, see bottom of page 167, they can be computed in terms of the metric coeffiencts g_{ij}. It is a very good exercise to deduce the important formula (4.6) on page 168. A variation of the definition of ∇ is the Levi-Civita covariant derivative along a curve. This leads to the concept of a parallel vector field along a curve (Definiton 4.6), and it is easy to check that a curve is self-parallel if and only if it is a geodesic. The identity (4.8) for the ambient covariant derivative on $\mathbb{R}^n$ is a fancy way of formulating the fact that partial derivatives are on $\mathbb{R}^n$ are independent of order, the last term with the Lie bracket $[X, Y]$ compensates for the fact that the first two terms involve derivatives of X and Y. From (4.8) and the Gauss formula one finds the Gauss curvature equation (4.10) and the Codazzi-Mainardi equation (4.11). The Riemann curvature tensor $R(X, Y)Z$ is defined as the left hand side of the Gauss curvature equation and measures the failure of the covariant derivatives ∇_X and ∇_Y to commute. One should note: (i) even though it is defined as a combination of derivatives $R(X, Y)Z$ depends only on the pointwise values of its arguments (since this is true for the right hand side of the Gauss curvature equation), (ii) the curvature tensor depends only on the Riemannian metric on M and not directly on the imbedding $M \subset \mathbb{R}^n$ (since this holds for ∇), (iii) the Gauss curvature equation identifies the intrinsic curvature of M measured by R with the bending of M in $\mathbb{R}^n$ measured by the shape operator.

Further reading. Read *Section 4.4*, where an interpretation of the mean curvature as the first variation of area is found. In Definition 4.59 a hypersurface $M \subset \mathbb{R}^n$ is defined to be minimal if its mean curvature vanishes identically.

In the exercise session we looked at hyperbolic space as the hyperboloid in Minkowski space. The computation of its second fundamental form and its geodesics are similar to the same computations for the round sphere S^{n-1} in $\mathbb{R}^n$. A very nice discussion of these spaces can be found in the lecture notes by Christian Bär, pages 88-98: `http://geometrie.math.uni-potsdam.de/documents/baer/skripte/skript-DiffGeoErw.pdf`. In particular the shape operator is computed on the top of page 90, and the geodesics are found in Lemma 4.5.2.

Recommended exercises. Exercise 4.40 (page 166), Exercise 4.42 (page 168).

Lecture 8, October 24. In this lecture we discussed tensors and tensor fields on manifolds. Tensors are multilinear maps taking tangent vectors or 1-forms as arguments. An example is the Riemannian curvature tensor introduced in the last lecture. There are two equivalent constructions, the first with pointwise vectors or 1-forms as arguments, the second with vector or 1-form fields as arguments. In *Section 7.1* tensors as multilinear maps are introduced in a general algebraic setting, read this entire section very carefully! In *Section 7.2* tensors on a manifold are defined by applying the tensor construction to the individual tangent spaces to produce $T^r_s(T_pM)$. These vector spaces are the fibers of a vector bundle $T^r_s(TM)$, and a tensor field is a section of this vector bundle. The crucial ingredient here is that the change of coordinate maps for an atlas of M will give first a transformation rule for tangent vector components in different charts, and thereby a transformation rule for the components of a tensor. The important Exercise 7.28 gives an example of this. In *Section 7.3* tensor fields on M are defined by a direct global definition, they are defined to be elements of $T^r_s(\mathfrak{X}(M))$. In *Section 7.4* the two definitions are shown to be equivalent. This follows from the fact that a globally defined tensor field only depends on the point-wise values of its arguments, as formulated in Proposition 7.32.

Further reading. For more details on modules and multilinear algebra have a look at Appendix D, pages 649-662.

Recommended exercises. Exercise 7.11 (page 313), Exercise 7.16 (page 315), Exercise 7.28 (page 321), Exercise 7.33 (page 326).

Lecture 9, October 31. We continued the discussion on tensors with *Section 7.5* where the Lie derivative of tensor fields is constructed in two ways. The first construction (as a "tensor derivation") uses the algebraic operations of contraction and tensor product to extend the Lie derivative from functions and vector fields to general tensor fields. With this method one can also extend the Levi-Civita covariant derivative to general tensor fields. The second definition of the Lie derivative is as the derivative along the flow of a vector field, see (7.11). In *Section 7.6* metric tensors are introduced. At the moment a main reason to discuss metric tensors and (semi-)Riemannian metrics on a manifold is that they give type-changing isomorpism of tensors. This is discussed on pages 333-336. Differential forms are alternating covariant tensor fields. In *Section 8.1* the relevant linear algebra is discussed. Note in particular Lemma 8.6 where the wedge product is related to the determinant, and the Theorem 8.8 where a basis for $L^k_{\text{alt}}(V)$ is given. On page 353 the determinant of a linear map $\lambda \in L(V,V)$ is defined as the induced pull-back map on the 1-dimensional vector space $L^n_{\text{alt}}(V)$, where $n = \dim(V)$. Further, an orientation on V is defined to be an equivalence class of non-zero elements $\omega \in L^n_{\text{alt}}(V)$ where $\omega_1 \sim \omega_2$ if $\omega_1 = c\omega_2$ for $c > 0$. Or, an orientation is a a choice of component of $L^n_{\text{alt}}(V) \setminus \{0\}$.

Further reading. Have a look at Section 8.1.1 where alternating tensors are constructed as the Grassman algebra.

Recommended exercises. Exercise 7.54 (page 334), Exercise 8.15 (page 353).

Lecture 10, November 7. In *Section 8.2* the vector bundle $L^k_{\text{alt}}(TM) \to M$ are constructed using the standard procedure. The sections of this bundle are called differential k-forms on M and the space of such is denoted by $\Omega^k(M)$. For a map $f : M \to N$ there is a an induced pull-back map $f^* : \Omega^k(N) \to \Omega^k(M)$. Have a careful look at the formula on page 362 for pull-back in local coordinates, this involves "partial Jacobi determinants" of the map f. In the case $k = \dim M = \dim N$ it involves the (full) Jacobi determinant of f. In *Section 8.3* the exterior derivative d on forms is introduced. It is defined by its properties in Theorem 8.37, note in particular the expression in local coordinates. Read the proof of Lemma 8.39 which states that d commutes with pull-back.

Orientations are introduced in *Section 8.7.* In the book the case of general vector bundles $E \to M$ is treated, we need only consider the tangent bundle. Proposition 8.59 tells us that there is $\omega \in \Omega^n(M)$, $n = \dim M$, with $\omega_p \neq 0$ for all $p \in M$ if and only if there is an atlas for M for which all transition maps have positive determinants. An equivalence class $[\omega]$ of such ω gives an orientation of TM and of M. There seems to be some confusion in Definition 8.67: an oriented atlas should be an atlas for which all transition maps have positive determinant (the description in this definition is equivalent to a positively oriented atlas).

Further reading. Read the proof of Cartan's formula for the Lie derivative of differential forms, $\mathcal{L}_X = d \circ i_X + i_X \circ d$, in *Section 8.6.*

Recommended exercises. Problems (6), (11), page 388-389. (With integration of forms introduced in the next chapter we will have a tool to decide that ω is not exact.)

Lecture 11, November 21. Today we discussed integration of differential forms over oriented manifolds, as defined in the beginning of *Chapter 9.* For a manifold with an oriented atlas the integration of a form in local coordinates does not depend on the particular chart, this is the computation on page 392. In Definition the integral is defined using an oriented atlas and a partition of unity. Proposition 9.3 tells us that it is independent of all choices. For explicit computations it is usually impossible to compute with a partition of unity. By removing a subset of measure zero from the manifold one can compute the integral using disjoint charts and no partition of unity, see Theorem 9.7. In *Section 9.1* Stokes' Theorem is proved. This relates the integral to the exterior derivative by $\int_M d\omega = \int_{\partial M} \omega$. The proof of Stokes' theorem consists of first studying the special case where M is the euclidean half-space. The general case then follows by the definition of the integral through a sum of local contributions, together with the fact that the boundary has the orientation induced by the outward pointing normal vector field. See *Section 8.7.1* for the definition of the induced orientation on the boundary.

Further reading. We leave Chapter 9 after Stokes' Theorem, but most of the remaining sections are very interesting. In *Section 9.2* the divergence of a vector field is defined (it depends on a choice of volume form, for example as given by a Riemannian metric), and a divergence theorem is proved. In *Section 9.3* the topological implications of Stokes' theorem is discussed.

Recommended exercises. Problem (1), page 437. Further, show that ω is not exact (compare problem (11) on page 389.)

Lecture 12, November 28. In this lecture we first computed an explicit example illustrating Stokes' Theorem. Then we started the final section of this course which is *Chapter 13* on Semi-riemannian geometry. We are primarily interested in two special cases of semi-riemannian metrics. First the riemannian case where the inner product is positive definit (and defines a norm of vectors), second the Lorentzian case where the metric has one negative direction (which models a spacetime with one time dimension). In the beginning of Chapter 13 the casual type (spacelike/lightlike/timelike) of tangent vectors and curves are defined. In Definition 13.5 the length of a curve is defined, this definition is meaningful when the curve has a casual type. For curves in a riemannian manifold it has the ordinary meaning of length, for timelike curves in a Lorentzian manifold it means the proper time for an observer represented by the curve. In *Section 13.1* the Levi-Civita connection ∇ of a Riemannian metric is introduced. We have seen this before in Section 4.3, the difference here is that it is defined uniquely as a connection which is torsion free and respects the given semi-riemannian metric. The existence and uniqueness follows from the important Koszul formula in Theorem 13.9. Given a curve $c : I \to M$ we say that a vector field $X : I \to TM$ along c is parallel if $\nabla_{\partial/\partial t} X = 0$. This is an ordinary differential equation which has a unique solution $X(t)$ given initial data $X(0) = X_0 \in T_{c(t_0)}M$. The parallel transport map $P(c)^t_{t_0} : T_{c(t_0)}M \to T_{c(t)}M$ is defined by $P(c)^t_{t_0}(X_0) := X(t)$.

Further reading. Jump ahead and have a look at Section 13.6 on Lorenzian geometry.

Recommended exercises. Exercise 13.14 (page 552), Exercise 13.15 (page 553).

Lecture 13, December 5. We continued the discussion on Semi-Riemannian geometry by introducing the curvature tensor R, *Section 13.2*. The first geometric interpretation of R is that it is the obstruction for the manifold to be locally isometric to $\mathbb{R}^n_\nu$, see Theorem 13.18. For a more elementary proof of this theorem, see the online supplement to the course book, `http://www.ams.org/bookpages/gsm-107/Supp.pdf`, page 135. Very important are the symmetries of R, stated in Theorems 13.19 and 13.20. The sectional curvature K (page 557) contains the same information as R (Proposition 13.27). The Ricci curvature is a trace of R and is a symmetric 2-tensor (Definition 13.29). Note two misprints on page 559 in the book: first the Ricci tensor is a $(0,2)$-tensor, second $\mathrm{Ric} \geq k$ means that $\mathrm{Ric}(v,v) \geq k\langle v,v\rangle$ for all $v \in TM$.

In *Section 13.4* a curve $\gamma : I \to (M,g)$ is defined to be a geodesic if its tangent vector is parallel along the curve, $\nabla_{\dot\gamma}\dot\gamma = 0$. With a given initial vector $v \in T_pM$ there is a unique geodesic γ_v with $\dot{\gamma_v}(0) = v$ defined over a maximal interval, see Proposition 13.50. The exponential map is the smooth map which takes v to $\gamma_v(1)$, this is a diffeomorphism of a neighbourhood of $0 \in T_pM$ with a neighbourhood of $p \in M$. Choosing a basis of the vector space T_pM gives us normal coordinates around the point p, see page 572.

Further reading. Read *Section 13.3* on Semi-Riemannian submanifolds. The situation here is very similar to that in Chapter 4.

Recommended exercises. Exercise 13.54 (page 569), Exercise 13.57 (page 570).

Lecture 14, December 12. In *Section 13.5* the distance function on a Riemannian manifold is introduced, see (13.8). The distance between points p and q is the infimum of lengths of paths from p to q. In general this infimum need not be attained. Proposition 13.86 tells us that the distance to the center point in a normal coordinate system is attained and is given by the radial geodesics. In general, if the distance between points is attained by a curve, then Proposition 13.88 tells us that this curve must be a geodesic.

Jacobi fields are introduced in *Section 13.7*. They arise from the "linearization" of the equation for geodesics in (M,g). The equation for Jacobi fields (Def. 13.108) is a second order linear ordinary differential equation, so the set of solutions is a vector space and a solution is uniquely determined by initial data (Thm. 13.109). Theorem 13.110 says that the Jacobi fields along the geodesic γ split into tangential and normal parts. Conjugate points along a geodesic γ are introduced in Definition 13.116, and characterized in Theorem 13.117. Note that points p, q on γ are conjugate if and only if there is a non-trival variation of γ through geodesics which keeps the points p, q fixed to first order.

A main theme of Semi-Riemannian geometry is to use information about the curvature tensor to deduce statements about the geometry or the topology of the manifold. We jump to the end of *Section 13.8*, page 611-612, to find one result of this type. Proposition 13.137 tells us that a Riemannian manifold for which the sectional curvature is nowhere positive has no conjugate points on any geodesic.

Further reading. Have a look at Section 13.13 where a short introduction to General Relativity is given. In particular Jacobi fields are discussed in the section on tidal forces, page 630.

Recommended exercises. Problems (1) and (2), page 634.

Lecture 15, December 19. In the last lecture we discussed three classical theorems of Riemannian geometry from *Section 13.9*.

The Hopf-Rinow theorem (Theorem 13.139) states that different concepts of completeness are equivalent for a Riemannian manifold. A corollary is that any compact manifold is geodesically complete, so at any point the exponential map is defined on the whole of the tangent space at that point.

The Hadamard theorem tells us that for a manifold with non-positive sectional curvature the exponential map at any point is a local diffeomorphism and thus a covering map. This gives two conclusions. First, if the manifold is simply connected then the exponential map is a diffeomorphism so the manifold is diffeomorphic to euclidean space. Second, if the manifold is compact the fundamental group must be infinite. The proof of Hadamard's theorem uses Jacobi fields and the assumption on the sectional curvature rules out any conjugate points.

Myers's theorem gives a bound on the diameter for a manifold with a positive lower bound on its Ricci curvature. In particular such a manifold is compact. Since the same holds for the universal cover, this is also compact and the manifold must have finite fundamental group. The proof of Myers's theorem uses the second variation of the length functional on curves, see (13.9) page 604. One can also use the energy functional which gives slightly simpler computations, see Section 6.3 of the lecture notes by Bär, `http://geometrie.math.uni-potsdam.de/documents/baer/skripte/skript-DiffGeoErw.pdf`.