DIFFERENTIAL GEOMETRY, FALL 2011
GRADUATE LEVEL, ADDITIONAL HOMEWORK 1
LIE GROUPS

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Read Chapter 5 on Lie groups, then solve and hand in your solutions to the following three problems from this chapter. Please pay attention to the presentation as well as the arguments you give in your solutions.

Problem 1. Show that if we consider $\mathrm{SL}(2,\mathbb{R})$ as a subset of $\mathrm{SL}(2,\mathbb{C})$ in the obvious way, then $\mathrm{SL}(2,\mathbb{R})$ is a Lie subgroup of $\mathrm{SL}(2,\mathbb{C})$ and $\wp(\mathrm{SL}(2,\mathbb{R}))$ is a Lie subgroup of Mob. Show that if $T \in \wp(\mathrm{SL}(2,\mathbb{R}))$, then T maps the upper half-plane of $\mathbb{C}$ onto itself (bijectively).

Problem 2. Show that $\mathrm{SO}(3)$ is the connected component of the identity in $\mathrm{O}(3)$. Show that the special Lorentz group $\mathrm{SO}(1,3)$ is not connected. Show that the first entry of elements of $\mathrm{SO}(1,3)$ must have absolute value greater than or equal to 1. Define $\mathrm{SO}(3,1)^{\uparrow}$ as the subset of $\mathrm{SO}(1,3)$ consisting of matrices with positive first entry (which must be greater than 1). Show that $\mathrm{SO}(3,1)^{\uparrow}$ is connected (and hence the connected component of the identity in $\mathrm{O}(1,3)$).

Problem 3. Let $A \in \mathfrak{gl}(V) = L(V,V)$ for some finite-dimensional vector space V. Show that if A has eigenvalues $\{\lambda_i\}_{i=1,\ldots,n}$, then $\mathrm{ad}(A)$ has eigenvalues $\{\lambda_j - \lambda_k\}_{j,k=1,\ldots,n}$. Hint: Choose a basis for V such that A is represented by an upper triangular matrix. Show that this induces a basis for $\mathfrak{gl}(V)$ such that with the appropriate ordering, $\mathrm{ad}(A)$ is upper triangular.

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