DIFFERENTIAL GEOMETRY, FALL 2011
GRADUATE LEVEL, ADDITIONAL HOMEWORK 2
FIBER BUNDLES

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Read Chapter 6 on fiber bundles, then solve and hand in your solutions to the following four problems from this chapter. Please pay attention to the presentation as well as the arguments you give in your solutions.

Problem 1. Construct a principal $\mathbb{Z}_2$-bundle P and left actions λ_1 and λ_2 on S^1 and $\mathbb{R}$ respectively, such that $P \times_{\lambda_1} S^1$ is the twisted torus and $P \times_{\lambda_2} \mathbb{R}$ is the Möbius band line bundle.

Problem 2. Let $\xi = (E, \pi, M, F)$ be a G-bundle. Let $g_{\alpha\beta}$ be cocycles associated to a G-atlas $\{(U_\alpha, \varphi_\alpha)\}$ for ξ. Show that ξ is G-equivalent to a product bundle if and only if there exist functions $\lambda_\alpha : U_\alpha \to G$ such that $g_{\beta\alpha}(x) = \lambda_\beta(x)\lambda_\alpha^{-1}(x)$ for all $x \in U_\alpha \cap U_\beta$ and all α, β.

Problem 3. Show that the twisted torus of Example 6.17 is trivial as a fiber bundle but not trivial as a $\mathbb{Z}_2$-bundle.

Problem 4. Show that the tangent bundle of the real projective space $\mathbb{R}P^n$ is a vector bundle isomorphic to $\mathrm{Hom}(\mathbb{L}(\mathbb{R}P^n), \mathbb{L}(\mathbb{R}P^n)^\perp)$, where $\mathbb{L}(\mathbb{R}P^n) \to \mathbb{R}P^n$ is the tautological line bundle and $\mathbb{L}(\mathbb{R}P^n)^\perp \to \mathbb{R}P^n$ is the rank n vector bundle whose fiber at $l \in \mathbb{R}P^n$ is $\{(l, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \perp l\}$.

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