DIFFERENTIAL GEOMETRY, FALL 2011
GRADUATE LEVEL, ADDITIONAL HOMEWORK 3
DISTRIBUTIONS AND FROBENIUS' THEOREM

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Read Chapter 11 on distributions and Frobenius theorem. Most important are sections 11.1-11.4, but have a look also at the sections 11.5-11.7. Then solve and hand in your solutions to the following three problems from this chapter. Please pay attention to the presentation as well as the arguments you give in your solutions.

Problem 1. (Problem (1), page 498.) Show that the following vector fields define a rank 2 distribution on $\mathbb{R}^3$ which is not involutive (and hence not integrable):

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.$$

Draw a picture of the portion of this distribution which sits at points in the (x, y)-plane and try to see geometrically why the distribution is not integrable.

Problem 2. (Problem (3), page 498.) Let θ be a 1-form. Show that a 2-form η is in the ideal generated by θ if and only if $\eta \wedge \theta = 0$.

Problem 3. (Problem (4), page 498.) Consider the system of partial differential equations

$$\frac{\partial z}{\partial x} = F(x, y, z),$$
$$\frac{\partial z}{\partial y} = G(x, y, z).$$

Show that the graphs of solutions to these equations are integral manifolds of the distribution defined by the 1-form $\theta = dz - Fdx - Gdy$. Use Theorem 11.18 to deduce integrability conditions for this system. (Hint: use the previous problem.)

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