DIFFERENTIAL GEOMETRY, FALL 2011
HOMEWORK 1

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Solve and hand in your solutions to the following seven problems (they are all from Chapter 2 in the course book). You are also welcome to solve the quite complicated bonus problem (this will help towards, but is not necessary for, getting a high grade). Please pay attention to the presentation as well as the arguments given in the solutions.

Problem 1. Recall that we have charts on $\mathbb{R}P^2$ given by

$$[x,y,z] \mapsto (u_1,u_2) = (x/z, y/z) \text{ on } U_3 = \{z \neq 0\},$$
$$[x,y,z] \mapsto (v_1,v_2) = (x/y, z/y) \text{ on } U_2 = \{y \neq 0\},$$
$$[x,y,z] \mapsto (w_1,w_2) = (y/x, z/x) \text{ on } U_1 = \{x \neq 0\}.$$

Show that there is a vector field on $\mathbb{R}P^2$ which in the last coordinate chart above has the coordinate expression $w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}$. What are the expressions for this vector field in the other two charts?

Problem 2. Show that the subset N of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ defined by $N = \{(x,y) : \|x\| = 1,\ x \cdot y = 0\}$ is a smooth manifold diffeomorphic to TS^n.

Problem 3. Find the integral curves and the flow of the vector field on $\mathbb{R}^2$ given by $X(x,y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

Problem 4. Using the usual spherical coordinates (φ, θ) on S^2, compute the bracket $[\varphi\frac{\partial}{\partial\theta}, \theta\frac{\partial}{\partial\varphi}]$.

Problem 5. Show that if X and Y are vector fields with flows φ_t^X and φ_t^Y, then if $[X,Y] = 0$, the flow of $X+Y$ is $\varphi_t^X \circ \varphi_t^Y$.

Problem 6. Recall that the tangent bundle of the open set $\mathrm{GL}(n,\mathbb{R})$ in $M_{n\times n}(\mathbb{R})$ is identified with $\mathrm{GL}(n,\mathbb{R}) \times M_{n\times n}(\mathbb{R})$. Find the flow of the vector field X on $\mathrm{GL}(n,\mathbb{R})$ given by $g \mapsto (g, g^2)$. Is X a complete vector field?

Problem 7. Show that if $p(x) = p(x_1, \ldots, x_n)$ is a homogeneous polynomial, so that for some positive integer m,

$$p(tx_1, \ldots, tx_n) = t^m p(x_1, \ldots, x_n),$$

then as long as $c \neq 0$, the set $p^{-1}(c)$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

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Bonus Problem. Let

$$t \mapsto Q_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $t \in \mathbb{R}$. Let $\varphi(t, P) = Q_t P$ where P is a plane in $\mathbb{R}^3$ (containing the origin). Show that φ defines a flow on the Grassmann manifold $G(3, 2)$. Find the local expression in some coordinate system of the vector field X^Q generating this flow. Do the same thing for the flow given by

$$t \mapsto R_t = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix}$$

and find the corresponding vector field X^R. Compute the bracket $[X^R, X^Q]$.