DIFFERENTIAL GEOMETRY, FALL 2011
HOMEWORK 2

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Solve and hand in your solutions to the following four problems (where most are from Chapters 3 and 4 in the course book). You are also welcome to solve the more complicated bonus problem (this will help towards, but is not necessary for, getting a high grade). Please pay attention to the presentation as well as the arguments given in the solutions.

Problem 1. Let the function $s : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be defined by the rule that $s(x)$ is the line through x and the origin. Show that s is a submersion.

Problem 2. Show that the map $f : \mathbb{R}P^2 \to \mathbb{R}^3$ defined by $f([x, y, z]) = \frac{1}{r^2}(yz, xz, xy)$, $r^2 = x^2 + y^2 + z^2$, is an immersion at all but six points $p \in \mathbb{R}P^2$. The image of f is called the Roman surface. Show that the map $g : \mathbb{R}P^2 \to \mathbb{R}^4$ given by $g([x, y, z]) = \frac{1}{r^2}(yz, xz, xy, x^2 + 2y^2 + 3z^2)$ is a smooth embedding.

Problem 3. Find the Gauss curvature K at a point (x, y, z) on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Problem 4. Show that the vector field $X(x, y, z) = y\partial/\partial x - x\partial/\partial y$ is tangential to the unit sphere $S^2 \subset \mathbb{R}^3$. Compute $\nabla_{Y_p} X$ for any tangent vector $Y_p \in T_p S^2$, where ∇ denotes the Levi-Civita covariant derivative of S^2. Show that X satisfies the identity

$$g(\nabla_{Y_p} X, Z_p) + g(Y_p, \nabla_{Z_p} X) = 0$$

for all $Y_p, Z_p \in T_p S^2$. (This is the definition of X being a *Killing vector field*, which means that the diffeomorphisms in the flow generated by X are all isometries of the Riemannian metric on S^2.)

Bonus Problem. Enneper's surface. Show that the following defines a minimal but not 1-1 immersion.

$$x(u, v) := \left(u - \frac{u^3}{3} + uv^2, -v + \frac{v^3}{3} - vu^2, u^2 - v^2 \right).$$

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