DIFFERENTIAL GEOMETRY, FALL 2011
HOMEWORK 3

MATTIAS DAHL

Solve and hand in your solutions to the following five problems covering material from Chapters 7-9 in the course book. You are also welcome to solve the more complicated bonus problem (this will help towards, but is not necessary for, getting a high grade). Please pay attention to the presentation as well as the arguments given in the solutions.

Problem 1. (Problem (7), page 343.) Let the diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $(x, y) \mapsto (x + 2y, y)$. Let $\tau = x\frac{\partial}{\partial x} \otimes dy + \frac{\partial}{\partial y} \otimes dy$. Compute $\varphi_* \tau$ and $\varphi^* \tau$.

Problem 2. (Problem (12), page 343.) In some chart $(U, (x, y))$ on a 2-manifold, let $A = x\frac{\partial}{\partial y} \otimes dx \otimes dy + \frac{\partial}{\partial x} \otimes dy \otimes dy$ and let $X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Compute the coordinate expression for the Lie derivative $\mathcal{L}_X A$.

Problem 3. Let $\omega = x^2 dy \wedge dz + y^2 dz \wedge dw \in \Omega^2(\mathbb{R}^4)$ and let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by $f(a, b, c) = (a, b, c, abc)$. Compute $f^* \omega$.

Problem 4. Let the 2-torus $T^2 = S^1 \times S^1$ be embedded in $\mathbb{R}^4$ as $T^2 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + w^2 = y^2 + z^2 = 1\}$. Give T^2 an orientation and compute $\int_{T^2} \omega$ where $\omega = xyzdw \wedge dy \in \Omega^2(\mathbb{R}^4)$.

Problem 5. Find a differential form $\eta \in \Omega^{n-1}(\mathbb{R}^n)$ such that $\mathrm{vol}(U) = \int_{\partial U} \eta$ for all bounded domains $U \subset \mathbb{R}^n$ with smooth boundary. Find an expression for all such forms η.

Bonus Problem. Let M be a smooth manifold with a Riemannian metric, and let $SM \subset TM$ be the subset of tangent vectors of unit length. Show that SM is an orientable smooth manifold (regardless of whether M is orientable or not).

Date: November 22, 2011.