DIFFERENTIAL GEOMETRY, FALL 2011
HOMEWORK 4

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Solve and hand in your solutions to the following three problems covering material from Chapter 13 in the course book. You are also welcome to solve the more complicated bonus problem (this will help towards, but is not necessary for, getting a high grade). Please pay attention to the presentation as well as the arguments given in the solutions.

Problem 1. Prove that the Ricci tensor Ric of a semi-Riemannian manifold (M, g) is symmetric, that is $\mathrm{Ric}(v, w) = \mathrm{Ric}(w, v)$.

Problem 2. (Exercise 13.30, page 560.) Show that if M is connected and $\dim(M) > 2$, and $\mathrm{Ric}(\cdot, \cdot) = f\langle \cdot, \cdot \rangle$, where $f \in C^\infty(M)$, then $f \equiv k$ for some $k \in \mathbb{R}$ (so (M, g) is Einstein).

Problem 3. (Problem (9), page 635.) Let $H := \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$ be the upper half-plane endowed with the metric

$$g := \frac{1}{v^2}(du \otimes du + dv \otimes dv).$$

Show that H has constant sectional curvature $K = -1$. Find which curves are geodesics.

Bonus Problem. (The Clifton-Pohl torus.) Let $M := \mathbb{R}^2 \setminus \{0\}$ be equipped with the Lorentzian metric

$$g = \frac{1}{u^2 + v^2}(du \otimes dv + dv \otimes du).$$

For any $c \neq 0$ the map $(u, v) \mapsto (cu, cv)$ is an isometry of (M, g). Let the group $\mathbb{Z}$ act on M by isometries by letting 1 act as $\mu : (u, v) \mapsto (2u, 2v)$ and set $T = M/\mathbb{Z}$. Topologically T is the annulus $\{1 \leq r \leq 2\}$ with boundary points identified by μ, in particular it is compact.

Show that the curve $\gamma(t) = ((1 - t)^{-1}, 0)$, $-\infty < t < 1$, is a geodesic in (T, g) which cannot be extended to a larger interval of t. (This example shows that a compact Lorentzian manifold need not be geodesically complete.)

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