1 Introduction

1.1 Integration and boundaries

We shall start with some well-known results on integration which we then shall try to interpret in a way that leads to some of the basic ideas of algebraic topology. The basic result that we are going to discuss is Stokes formula[1] but we shall also spend some time in formulating it in a way that will suit our purposes.

Remark: Note that, as with all results of analysis, our integration formulas require various conditions on the functions involved to be true. As that is not our main concern we shall shrug them all under the mat and assume that our functions fulfil all conditions necessary to have our formulas make sense (and be true). (Assuming that they are smooth with compact support will certainly ensure that.)

1.1.1 Curve integrals

We start with a very simple situation namely that of the fundamental theorem of integral calculus:

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Some comments are in order.

- We integrate over the interval $[a, b]$ and the right hand side involves an evalutation at its boundary points a and b.
- On the right hand side a and b are treated differently, the evaluation at b appearing with a plus sign, that at a with a minus sign. On the left hand side this is reflected in the fact that we integrate in a certain direction, from a to b.

The ideas that are relevant to us can be made clearer of we pass to curve integrals (in the plane say). Hence we start with a curve γ in $\mathbf{R}^2$ and we want to integrate along it (the interval $[a, b] \subset \mathbf{R} \subset \mathbf{R}^2$ is one such example in which case we should get the usual integral). The first question is what form the integrand should have. We can take our cue from the definition of the ordinary integral in terms of Riemann sums. Hence if we subdivide the interval $a = t_0 < t_1 < \cdots < t_n = b$ and pick points $t_{i-1} \leq v_i \leq t_i$ then the Riemann sum is $\sum_{i=1}^n f(v_i)\Delta t_i$ where $\Delta t_i = t_i - t_{i-1}$. If we try to generalise this to a curve integral we see that by a curve γ we must mean something with the property that given two distinct points p and q on it we can speak of p coming before q (written $p < q$) or vice versa and exactly one of these must be true. Assuming that we can pick points $p_o < p_1 < \cdots < p_n$ on γ and points also on the curve $p_{i-1} \leq v_i \leq p_i$. The difference with the previous case is that we have two possible differences that can be used in a Riemann sum; the difference $\Delta x_i := x_i - x_{i-1}$ of the x-coordinates of p_i and p_{i-1} and $\Delta y_i := y_i - y_{i-1}$ using instead the y-coordinates. This suggests that we should use two functions f and g in the integrand and consider the Riemann sum $\sum_{i=1}^n f(v_i)\Delta x_i + g(v_i)\Delta y_i$. When these sums converge to a common value it seems reasonable to denote it $\int_\gamma f(x, y)\,dx + g(x, y)\,dy$.

There is one important special case; when the curve is parametrised, i.e., γ is a map $\gamma\colon [a, b] \to \mathbf{R}^2$. To begin with this solves the problem on how to order the points, we use the parameter value. Doing that it is not difficult to show that, with $\gamma(t) = (x(t), y(t))$,

$$\int_\gamma f(x, y)\,dx + g(x, y)\,dy = \int_a^b (f(x(t), y(t))x'(t) + g(x(t), y(t))y'(t))\,dt.$$

[1] For definition see `http://en.wikipedia.org/wiki/Stokes_theorem`.

At least mnemonically this can be expressed by the formulas $dx = x'(t)\,dt$ and $dy = y'(t)\,dt$. The answer is that if we define, for a function h defined in a neighbourhood of γ, $dh := h'_x\,dx + h'_y\,dy$ then

$$\int_\gamma dh = h(\gamma(b)) - h(\gamma(a))$$

where we have assumed that the curve is parametrised as above.

Exercise 1: Show this formula using the formula above for the the curve integral along a parametrised curve.

Note that not every expression $f(x,y)\,dx + g(x,y)\,dy$ is of the form dh. In fact if $f(x,y) = h'_x(x,y)$ and $g(x,y) = h'_y(x,y)$, then $g'_x = h''_{yx} = h''_{xy} = f'_y$ which is not always true. We shall come back to this question.

To prepare for going further we now introduce some notation. First we shall introduce formal linear combinations, with real coefficients, of points in the plane; $\sum_i c_i[z_i]$ where $c_i \in \mathbf{R}$ and $z_i \in \mathbf{R}^2$. By "formal" we mean that addition and scalar multiplication is in now way related to the vector space structure of $\mathbf{R}^2$ (and we use $[z]$ instead of z to emphasise this) and that we have no linear relation between these elements (i.e., the set $\{[z] \mid z \in \mathbf{R}^2\}$ form a basis). For reasons that will become clearer later we shall call such linear combinations 0-*cycles* (in $\mathbf{R}^2$ this clearly makes sense for any set and we shall soon consider a subset of $\mathbf{R}^2$ and consider 0-cycles in that set). If $f\colon \mathbf{R}^2 \to \mathbf{R}$ then we extend it "by linearity" to 0-cycles; $f(\sum_i c_i[z_i]) := \sum_i c_i f(z_i)$. Hence we can express our integration formula as $\int_\gamma dh = h([b] - [a])$. Furthermore, a and b are the boundary points of γ and we formalise this by putting $\partial\gamma = [b] - [a]$. Note tha asymmetric treatment of a and b; γ comes with a direction (or as we shall say *orientation*) and this allows us to define an *oriented* boundary of it where we assign different signs to the beginning and end point. We can then once again rewrite our formula as $\int_\gamma dh = h(\partial\gamma)$. Going one step further (which would in fact seem to be going too far) we can write evaluation on a 0-cycle $z = \sum_i c_i[z_i]$ as an integral, $\int_z f = f(z) = \sum_i c_i f(z_i)$, and then the formula becomes $\int_\gamma dh) \int_{\partial\gamma} h$. This will turn out not to be as silly as it seems.

1.1.2 Curve and surface integrals

We shall now introduce one further step involving 2-dimensional integration. It will turn out to be very convenient to go one step further in our integral notation. We shall thus introduce 1-*chains* which are formal linear combinations of curves, $\gamma = \sum_i c_i[\gamma_i]$ (as it is less likely to introduce confusion we shall often here dispense with the brackets) and extend integration again by linearity;

$$\int_\gamma f(x,y)\,dx + g(x,y)\,dy := \sum_i c_i \int_{\gamma_i} f(x,y)\,dx + g(x,y)\,dy$$

We shall now use integration over a domain in $\mathbf{R}^2$, where by a *domain* D we mean a compact subset which is the closure of its interior and whose boundary consists of (nice enough to allow for curve integration) some curves. It is usually not made clear as long as one talks of integration in euclidean space but just as integration along curves requires that they be oriented the same is true for integration over a domain. (This can be ignored in the case of domains as euclidean space has a canonical orientation.) In the plane this amounts to specifying what is left and right or rather what is counte clockwise rotation. If we orient the boundary curves of D then such an orientation may be compatible with the orientation of the domain which means that going forward along the curve is the same as moving counter clockwise. For the domain D we then define its *signed boundary*, as a 1-cycle $\partial D = \sum_i \pm[\gamma_i]$, where γ_i runs over the boundary curves and we choose the $+$-sign of the orientations are compatible and the $-$-sign if they are not. This is illustrated by Fig. 1 where the outer boundary γ_1 has a compatible orientation and the inner boundary γ_2 does not so that $\partial D = \gamma_1 - \gamma_2$. Note that the things we integrate over D are expressions of the form $f(x,y)\,dxdy$ and one may ask if there is some special form for such an expression

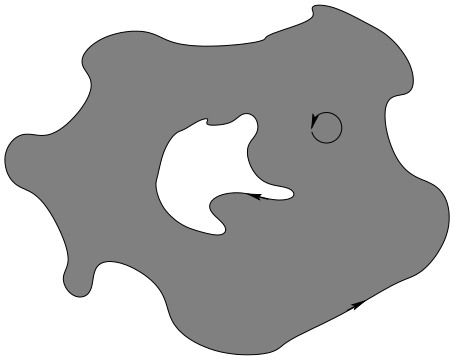

Figure 1: Oriented domain.

which will allows us to express this integration as an integral along the (oriented) boundary. By analogy with the curve integral case we should start with an expression $\omega = f(x,y)\,dx + g(x,y)\,dy$ and somehow construct an expression $d\omega$ and then hope that $\int_D d\omega = \int_{\partial D}\omega$. Such a d can be constructed with the following properties.

- d is linear so that $d\omega = d(f\,dx) + d(g\,dy)$.
- We have that $d(f dx) = df \wedge dx$ (and $d(g dy) = dg \wedge dy$).
- As before $df = f'_x dx + f'_y dy$ and $dg = g'_x dx + g'_y dy$.
- The product $\wedge$ is distributive so that $df \wedge dx = f'_x dx \wedge dx + f'_y dy \wedge dx$ and $dg \wedge dy = g'_x dx \wedge dy + g'_y dy \wedge dy$.
- It is also *anti-commutative* so that $dy \wedge dx = -dx \wedge dy$ and $dx \wedge dx = -dx \wedge dx$ which gives $dx \wedge dx = 0$.
- Together this gives $d\omega = (g'_x - f'_y) dx \wedge dy$.
- For the purposes of integration $dx \wedge dy$ is the same as $dxdy$ so that $\int_D d\omega = \int_D (g'_x - f'_y) dxdy$. Note that this means that $dy \wedge dx$ is *not* the same as $dydx$. In fact according to the conventions of integration $dydx$ is the same as $dxdy$ and by our previous rules $dy \wedge dx = -dx \wedge dy$ which then corresponds to $-dxdy$.
- It is now a fact that
$$\int_D d\omega = \int_{\partial D} \omega.$$

Remark: i) It may seem strange that $\int_D h(x,y)\,dx \wedge dy = \int_D h(x,y)\,dxdy$ whereas $\int_D h(x,y)\,dy \wedge dx = -\int_D h(x,y)\,dxdy$. However this has to do with the fact that the coordinates (x,y) is positively oriented, rotating the positive x-axis into the positive y-axis is moving counter-clockwise while (y,x) is negatively oriented, we move instead clockwise. Hence the $\wedge$-product keeps track of orientations.

ii) We seem to have the same notation for two different things, we use the symbols dx and dy but we have also defined dh for a function h. However, x and y are (linear) functions on $\mathbf{R}^2$, $x(s,t) = s$ and $y(s,t) = t$ and using our definition we should have $dx = x'_x\,dx + x'_y\,dy = 1 \cdot dx + 0 \cdot dy = dx$ so the two notations are consistent.

The whole pattern now persists in higher dimensions. A *k-form* in $\mathbf{R}^n$ is an expression of the form

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} f_{i_1, i_2, \cdots, i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, R$$

where the $f_{i_1,i_2,\cdots,i_k}$ are functions on $\mathbf{R}^n$. We then define $d\omega$ as

$$d\omega = \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} df_{i_1,i_2,\cdots,i_k} \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where we then use (obvious) generalisations of the rules above to get a $k+1$-form. A $k+1$-form can then be defined over a $k+1$-dimensional oriented compact subset with nice boundary[2] D of $\mathbf{R}^n$ and we have the formula $\int_D d\omega = \int_{\partial D} \omega$.

We can furthermore replace $\mathbf{R}^n$ with any open subset of $\mathbf{R}^n$.

Example: Let us consider the following question: Is the unit circle, S^1, the boundary of a domain (or more generally 2-chain) in $\mathbf{R}^2 \setminus \{0\}$? Of course it is the boundary of the unit disc in $\mathbf{R}^2$ but the unit disc contains the origin so it does not give an answer to our question. Assume now that $[S^1] = \partial D$ for a 2-chain D. We shall get a contradiction by integrating the 1-form dz/z. This requires some explanation. We identify $\mathbf{R}^2$ with the complex numbers $\mathbf{C}$. We also consider complex-valued functions and forms we cause no problem (this means that the $f_{i_1,i_2,\cdots,i_k}$ are complex-valued). We use z to denote the complex valued function on $\mathbf{R}^2 = \mathbf{C}$ which is just the identity function. In real coordinates we have $z = x + iy$ and thus $dz = dx + idy$. On the other hand z as a function on $\mathbf{C} \setminus \{0\}$ is invertible, i.e., $1/z$ makes sense. We can then express this in the form used above to define the notion of 1:

$$\frac{dz}{z} = \frac{x-iy}{|z|^2}(dx+idy) = \frac{x-iy}{|z|^2}\,dx + \frac{y+ix}{|z|^2}\,dy$$

However it is better to the original form. We begin by computing $\int_{S^1} \frac{dz}{z}$ (with S^1 oriented counter-clockwise). We do this by using the parametrisetion $t \mapsto e^{it} = \cos t + i \sin t$[3] where $t \in [0, 2\pi]$. This, together with the fact that the derivative of e^{it} equals ie^{it} gives

$$\int_{S^1} \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i.$$

The import thing is that this is non-zero because on the other hand

$$\int_{S^1} \frac{dz}{z} = \int_D d\left(\frac{dz}{z}\right)$$

and it is easy to see that $d(1/z) = -1/z^2 dz$ and thus

$$d\left(\frac{dz}{z}\right) = \frac{-1}{z^2} dz \wedge dz = 0.$$

Exercise 2: i) Verify that $d(1/z) = -1/z^2 dz$.

ii) Show that if ω is a 1-form (in any dimension), then $\omega \wedge \omega = 0$.

The conclusion then is that $\int_D d\left(\frac{dz}{z}\right) = 0$ and thus we get the contradiction $2\pi i = 0$.

1.1.3 Cycles and boundaries

In order to understand this let us introduce some notation. A k-form ω is *exact* if $d\omega = 0$ and it is *closed* if there is a $k-1$-form η such that $\omega = d\eta$. On the other hand a k-chain D is called a *cycle* if $\partial D = 0$ and it is a *boundary* if there is a $k+1$-chain S such that $\partial S = D$. It is now a fact that $d^2\omega = 0$ for all forms ω.

Exercise 3: Prove that $d^2\omega = 0$ for any form ω.

[2] What all this means is of course something we haven't defined properly.

[3] For definition see `http://en.wikipedia.org/wiki/Euler's_formula`.

On the other hand it seems clear that a boundary should itself have an empty boundary, i.e., $\partial^2 D = 0$ for all chains D. As we are working with signed boundaries it may ve less clear that the signs actually are the right ones to make this true but the following example should make this plausible.

Example: Consider a triangle T (cf., Fig. 2) where we have (as is natural) divided its boundary into three edges (a, b and c) and three vertices (x, y and z). With counter-clockwise orientation

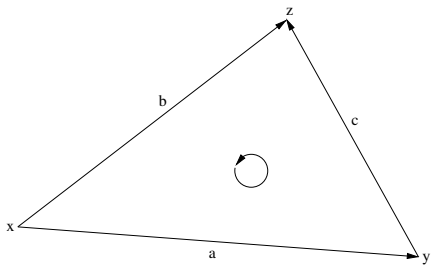

Figure 2: Simplex with edges and vertices.

and orientation of the edges as is indicated in the picture we have $\partial T = a - b + c$. Again with the indicated orientations we have $\partial a = y - x$, $\partial b = z - x$ and $\partial c = z - y$. Hence $\partial^2 T = (y - x) - (z - x) + (z - y) = 0$ so the signs do indeed cancel.

As a consequence we get that all exact forms are closed and all boundaries are cycles. It is then natural to ask the following two questions:

- Is every closed form exact and if not what is more precisely the difference?
- Is every cycle a boundary and if not what is more precisely the difference?

In the two cases the closed forms (cycles) form a sub-vector space of all forms (chains) and the exact forms (boundaries) is a subspace. Hence one way of measuring the difference is to take the quotient of these two vector spaces (and hence that quotient is zero precisely when the answer to the question is affirmative). This leads to the definition of the *de Rham cohomology groups*,[4] $H^k_{dR}(X/\mathbf{R})$, defined as

$$H^k_{dR}(X/\mathbf{R}) := \frac{\{\omega \ k\text{-form} \mid d\omega = 0\}}{\{d\eta \mid \eta \ k-1\text{-form}\}},$$

where so far we assume that X is an open subset of some $\mathbf{R}^n$ and the *homology groups*, $H_k(X, \mathbf{R})$, defined as

$$H_k(X, \mathbf{R}) := \frac{k\text{-cycles}}{k\text{-boundaries}}.$$

Stokes formula then gives a relation between these two vector spaces. Namely every k-form ω gives rise to a linear map on k-chains given by $D \mapsto \int_D \omega$.

Remark: So far we have been very cavalier about what we would accept as chains. As we see here one must be careful as in general integration even along a curve is not always defined, the curve may be non-rectifiable[5]. However, for purposes outside of integration even non-rectifiable curves may be used. We shall continue for a while to sweep these problems under the rug (the end result, the homology groups, turn out to be the same for many different choices of specification).

If ω is closed then Stokes formula says that this map vanishes on boundaries; $\int_{\partial S} \omega = \int_S d\omega = 0$. Hence a closed form gives rise to a linear map on $H_k(X, \mathbf{R})$. On the other hand if ω is exact

[4] Georges de Rham, 1903–1990

[5] For definition see `http://en.wikipedia.org/wiki/Rectifiable_curve`.

then it gives rise to the zero linear map on cycles; $\int_D d\eta = \int_{\partial D} \eta = 0$. This means that we get a linear map

$$H^k_{dR}(X/\mathbf{R}) \to \mathrm{Hom}_{\mathbf{R}}(H_k(X,\mathbf{R}),\mathbf{R}).$$

The so called de Rham's theorem says that this is an isomorphism.

Example: Let ω be a 1-form on the open connected subset $X \subseteq \mathbf{R}^n$ and assume that integration of ω vanishes on all 1-cycles. We now try to find a function h on X such that $\omega = dh$. The idea is that h should be a kind of primitive function of ω and we usually construct primitive functions by integration. We thus fix a point $x_0 \in X$ and try to define h by $h(x) := \int_\gamma \omega$ where γ is a curve in X starting at x_0 and ending at x. Such a curve exists as X is connected (and open). There is of course an immediate problem in that this may not be well-defined, it may depend on the curve. However if γ' is another such curve we have

$$\int_\gamma \omega - \int_{\gamma'} \omega = \int_{\gamma-\gamma'} \omega = 0,$$

the last equality as $\gamma - \gamma'$ is a cycle; $\partial(\gamma - \gamma') = \partial\gamma - \partial\gamma' = [x] - [x_0] - ([x] - [x_0]) = 0$ and ω is assumed to vanish on cycles. Hence h is a well-defined function and what remains to be shown is that if $\omega = \sum_i f_i dx_i$, then $h'_{x_i} = f_i$. To compute that the i'th partial derivative of h we should consider $(h(y) - h(x))/\epsilon$, where $y = x + (0,\dots,\overset{i}{\epsilon},\dots,0)$. We can compute $h(y)$ by letting γ' be the curve that starts with the fixed curve γ from x_0 to x and then continues to y with the straight line ℓ from x to y. However, that means that $h(y)$ is equal to $h(x)$ plus the integral $\int_\ell \omega$. We can parametrise ℓ by $t \mapsto x + (0,\dots,\overset{i}{t},\dots,0)$ where t runs from 0 to ϵ. This gives that $\ell'(t) = (0,\dots,\overset{i}{1},\dots,0)$ and hence that

$$\frac{h(y) - h(x)}{\epsilon} = \frac{1}{\epsilon}\int_\ell \omega = \frac{1}{\epsilon}\int_0^\epsilon f_i(x + (0,\dots,\overset{i}{t},\dots,0))\,dt$$

and when $\epsilon \to 0$ the last expression converges to $f_i(x)$.

Exercise 4: Extend this argument to the case when X is not assumed to be connected.

Hint: X is the disjoint union of its connected components which all are open.

Furthermore, in nice cases $H_k(X,\mathbf{R})$ is actually finite dimensional. Then, by de Rham's theorem so is $H^k_{dR}(X/\mathbf{R})$. Another consequence is that the map $H_k(X,\mathbf{R}) \to \mathrm{Hom}_{\mathbf{R}}(H^k(X,\mathbf{R}),\mathbf{R})$ also given by integration, $D \mapsto (\omega \mapsto \int_D \omega)$, is again an isomorphism so that knowledge of $H^k(X,\mathbf{R})$ is completely equivalent to knowledge of $H_k(X,\mathbf{R})$. One also gets an invariant of X, k'th *Betti number*, $b_k := \dim_{\mathbf{R}} H_k(X,\mathbf{R})$.

Exercise 5: Recall that the Cantor set[6] C is the closed set $\{\sum_{n\geq 1} a_n/3^n \mid a_n = 0,2\}$ and put $X := \mathbf{R}^2 \setminus C$. Show that $H_1(X,\mathbf{R})$ and $H^1_{dR}(X/\mathbf{R})$ are infinite dimensional.

Hint: Consider the cycles given by circles with center in C and radii such that they are disjoint from C and the forms $dz/(z-a)$ where $a \in C$ and pair them against each other.

1.2 Pseudomanifolds

Forms are very much tied to the real numbers; only forms with real (or complex) coefficients make sense in general. However when it comes to chains there is no reason to restrict oneself to real linear combinations. In fact there are other interesting choices. The choice that gives most information is to use integer coefficients but another interesting choice is to use $\mathbf{Z}/2$-coefficients. The reason why this is particularly interesting is that if S is a set and $\sum_i a_i[s_i]$ is a (finite)

[6]For definition see `http://en.wikipedia.org/wiki/Cantor_set`.

formal linear combination of elements of S with coefficients in $\mathbf{Z}/2$, then such a sum is completely specified by the set of s_i for which $a_i = 1$ and any finite subset of S appears in this way. Thus we get something more set-theoretic and the boundary ∂D of a chain D should really be just the set-theoretic boundary. In particular we should have no need for orientations.

We shall now go through an example where this will be made more explicit. We start by making specific what we in that example mean by chain. For this we start by defining the *standard n-simplex* as $\Delta^n := \{(x_0, x_1, \ldots, x_n) \in \mathbf{R}^{n+1} \mid x_i \geq 0, \sum_i x_i = 1\}$.

We may project onto the n last coordinates. As $x_0 = 1 - \sum_{i=1}^n x_i$ we get that this projection is injective and the image is $\{(x_i) \in \mathbf{R}^n \mid x_i \geq 0, \sum_i x_i \leq 1\}$. This allows us us to get a clear picture of what Δ^n looks like.

Example: i) Δ^{-1} is the empty set.
ii) Δ^0 is just the point 0.
iii) Δ^1 is the interval $[0, 1]$.
iv) Δ^2 is the triangle $\{(x, y) \mid x, y \geq 0, x + y \leq 1\}$.
v) Δ^3 is a tetrahedron $\{(x, y, z) \mid z, y, z \geq 0, x + y + z \leq 1\}$.

The reason why we don't define the simplex in this way is that we lose some symmetry by giving x_0 a special rôle. In particular if $S \subseteq \{0, 1, \ldots, n\}$ is a subset then we define the *face*, Δ^n_S, of Δ^n associated to S to be the subset defined by $x_i = 0$ when $i \in S$. By projecting onto the coordinates whose indices are not in S we get a homeomorphism between the face and the standard $n - |S|$-simplex (including $\emptyset = \Delta^{-1}$ when $S = \{0, 1, \ldots, n\}$). In particular when $S = \{i\}$ we get the i'th *facet* of Δ^n, also denoted Δ^n_i. We now choose our orientations so that the boundary if Δ^n is $\partial[\Delta^n] = \sum_{i=0}^n (-1)^n [\Delta^n_i]$. Note that we have identified Δ^n_i with the standard $n-1$-simplex so that $\partial[\Delta^n_i]$ is also defined (and becomes a sum of faces of Δ^n).

Remark: This is the orientation we have used in Fig. 2.

Exercise 6: Show that $\partial^2[\Delta^n] = 0$.

We can use these simplices to define what we mean by a triangulation.

Definition 1.1 *Let X be Hausdorff topological space. A* triangulation *of X consists of a collection of continuous maps $\{f_\alpha \colon \Delta^{n_\alpha} \to X\}$ fulfilling the following conditions.*

- *Each f_α is injective.*
- *Each point of X lies in the image of some f_α and two f_α do not have the same image.*
- *For an α and a subset $S \subseteq \{0, 1, 2, \ldots, n_\alpha\}$ the composite of the identification $\Delta^{n_\alpha - |S|} \to \Delta^{n_\alpha}_S$ and the restriction $f_{\alpha|\Delta^{n_\alpha}_S} \colon X \to$ is of the form f_β for some β. It is necessarily unique by the previous condition and we shall use the notation (α, S) for that β.*
- *For each choice of distinct indices α and β there are subsets $S \subseteq \{0, 1, 2, \ldots, n_\alpha\}$ and $T \subseteq \{0, 1, 2, \ldots, n_\}$ such that $(\alpha, S) = (\beta, T)$ and $f_{(\alpha,S)} = f_{(\beta,T)}$.*
- *A subset Z of X is closed precisely when $f_\alpha^{-1}(Z)$ is closed in Δ^{n_α}.*

Remark: i) Set-theoretically X is determined by the relations $f_{(\alpha,S)} = f_{(\beta,T)}$ as the disjoint union over the index set of the f_α with identifications given by the relation $(\alpha, S) = (\beta, T)$. The last condition tells us that the topology of X is also determined by that relation (X has the quotient topology[7] of the disjoint union of the simplices).

ii) Note that each simplex comes with an identification with the standard simplex. In particular its set of vertices comes with an identification with $\{0, \ldots, n\}$. For homological purposes we only need to know the induced total order on the vertices. For that in turn it is enough for each edge (i.e., 1-simplex) to tell which of its vertices is larger than the other. We shall do that by drawing an arrow on each edge pointing from the smallest element to the largest (the condition that we get a total order on each simplex is that there is no oriented cycle).

[7]For definition see `http://en.wikipedia.org/wiki/Quotient_topology`.

Example: i) The circle may be triangulated in many different ways. Fig. 3 shows three such ways with two, three and four vertices. Note that the one with three vertices is homeomorphic to the triangulation of the boundary of the standard 2-simplex given by the faces (except for the simplex itself) of that simplex.

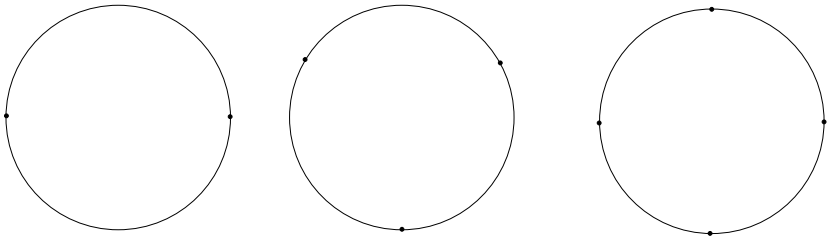

Figure 3: Triangulations of a circle.

ii) It is important that the intersection of simplices is a face of both simplices. Hence, the left hand figure of Fig. 4 is a triangulation while the right hand is not.

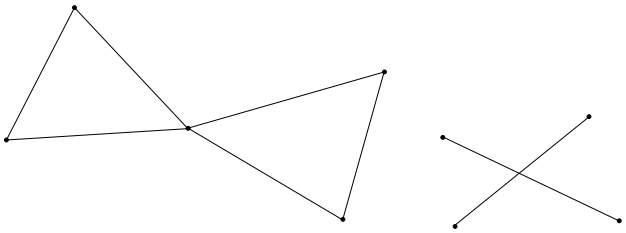

Figure 4: One triangulation and one non-triangulation.

Remark: As was mentioned before, if one consider triangulations up to homeomorphisms they are an essentially combinatorial concept. One common way of encoding such combinatorial information is through the notion of (abstract) *simplicial complex*. This concists of a set S and a collection $\mathcal{P}$ of finite subsets of S such that if $T \in \mathcal{P}$, then any subset of T is also in $\mathcal{P}$. The elements of $\mathcal{P}$ then corresponds to the simplices of the triangulation. This however gives a restriction; a simplex must be determined by the set of its vertices. This is not true for the first of the triangulations of Fig. 3 as there are two 1-simplices with the same vertices. It is easy enough to change the definition of simplicial complex so that it encompasses such examples (essentially one has to specify (α, S) rather than having it defined uniquely) but it becomes technically more complicated. Hence, particularly in combinatorics, simplicial complexes is often the preferred notion.

If X now has been provided with a triangulation $\{f_\alpha\}$ we can give a precise definition of a chain. Hence if R is a commutative ring a k-chain of X with coefficients in R is a formal finite linear combination $\sum_{n_\alpha=k} c_\alpha[\alpha]$ with $c_\alpha \in R$ (i.e., an element of the free R-module[8] on the set $\{\alpha \mid n_\alpha = k\}$). The R-module of k-chains we denote $C_k(X, R)$ (we suppress here the triangulation even though the same space may have different triangulations).The boundary map $\partial_k \colon C_k(X, R) \to C_{k-1}(X, R)$ is then extend by linearity from the boundary of simplices that we already defined for a standard simplex: $\partial_k[\alpha] = \sum_{i=0}^{n_\alpha}(-1)^i[(\alpha, \{i\})]$.

Exercise 7: Show that $\partial^2 = 0$.

[8]For definition see `http://en.wikipedia.org/wiki/Free_module`.

Example: Let us consider the example of the left hand side triangulation of the circle of Fig. 3. We have two 1-simpleces a and b and two vertices x and y. With suitable choice of orientations we have $\partial[a] = [x] - [y]$ and $\partial[b] = [y] - [x]$ and hence $[a] + [b]$ is a basis for 1-cycles and as there are no 2-chains we get that $H_1(S^1, R)$ is the free R-module $R([a] + [b])$. On the other hand all 0-chains are cycles and the 0-cycles are of the form $R([y] - [x])$. This gives that $H_0(S^1, R)$ is the free R-module $R\overline{[x]}$.

There is one ambiguity in this example. Formally the empty simplex is part of a triangulation (needed to account for the possibility that the intersection of two non-empty simplices could be empty). Hence we would also have -1-chains, all R-multiples of the single empty simplex and $\partial_0 \colon C_0(X, R) \to C_{-1}(X, R)$ which has $\partial_0[x] = [\emptyset]$ for all vertices x. Whether one wants to admit -1-chains or not depend on circumstances. However, as one may in fact be interested simultaneously in both cases one speaks of *reduced homology* when one does and uses the notation $\tilde{H}_k(X, R)$. Of course ordinary homology and reduced homology will only differ in degree 0, $H_0(X, R) = \tilde{H}_0(X, R) \bigoplus R$ (except in the *very* special case when $X = \emptyset$ and $\tilde{H}_{-1}(X, R) = R$ and all other homology modules are zero).

We are now able to arrive at a definition used in the title of this subsection.

Definition 1.2 *A* closed pseudomanifold *of dimension n is a topological space X together with a finite triangulation fulfilling the following conditions:*

- *There are no simplices of dimension larger than n and every simplex lies in an n-dimensional simplex.*
- *Every $n-1$-dimensional simplex lies in exactly 2 n-simplices*

Example: i) The circle with any of the triangulations of Fig. 3 is a pseudomanifold (the notion is actually independent of the triangulation).

The union of the boundary of two 3-simplices at one of their vertices is a pseudomanifold (see Fig. 5). It is not however a manifold, where a topological space is a manifold if each point has

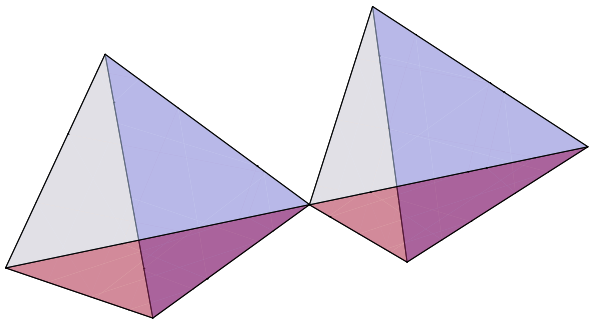

Figure 5: Non-manifold pseudomanifold.

a neighbourhood homeomorphic to an open subset of some $\mathbf{R}^n$. In fact the joined vertex has no such neighbourhood.

If we instead do the same construction for two 2-simplices, then the joined vertex lies on four 1-simplices and it is not a pseudomanfold.

ii) "Closed" comes from the condition that we have only finitely many simplices. For instance $\mathbf{R}$ is a (pseudo-)manifold with respect to the triangulation for which each interval $[n, n+1]$, $n \in \mathbf{Z}$ is a 1-simplex.

The reason for discussing pseudomanifolds here is that they have a very distinguished n-cycle.

Proposition 1.3 *Let X be a pseudomanifold of dimension n. Then the sum of all the n-simplices is a $\mathbf{Z}/2$-cycle.*

Proof. If we look at the sum $c = \sum_{n_\alpha=n}[\alpha]$ we want to compute $\partial c = \sum_{n_\alpha=n}\partial[\alpha]$. This of course is a sum (modulo 2) of $n-1$-simplices and the number of times a given $n-1$-simplex appears is equal to the times that it appears in some n-simplex (as it appears once or zero times in $\partial[\alpha]$ depending on whether it is contained in the simplex of α or not). However, by the definition of pseudomanifold any $n-1$-simplex appears in exactly two n-simplices which is an even number. □

The cycle of this proposition is called the *fundamental cycle* of the pseudomanifold. Most of the time it is the only non-zero cycle as the following exercise shows.

Exercise 8: Show that if the n-dimensional closed pseudomanifold X is connected then $H_n(X, \mathbf{Z}/2) = \mathbf{Z}/2\sigma$, where σ is the fundamental cycle.

It is natural to ask if there is a fundamental cycle in chains with other coefficients. It turns out that the integral case is the crucial one and this is the reason for the following definition.

Definition 1.4 *Let X be a pseudomanifold of dimension n. An* orientation *or integral* fundamental cycle *of X is an n-cycle $\sum_{n_\alpha=n} c_\alpha[\alpha]$ with integer coefficients, where $c_\alpha \in \{\pm 1\}$. We say that X is* orientable *if it admits an orientation.*

Example: i) The oriented boundary $\sigma = \partial[\Delta^n] = \sum_{0\leq i\leq n}(-1)^i[\Delta^n_i]$ is an $n-1$-chain of the boundary $\partial\Delta^n$ of Δ^n and each of its $n-1$-simplices appears with coefficient ± 1. We have seen that $\partial\partial[\Delta^n] = 0$ so that σ is an orientation of $\partial\Delta^n$.

ii) All three triangulations of the circle of Fig. 3 are orientable (as is easily seen).

Exercise 9: Show that if X is a closed connected n-pseudomanifold and $c = \sum_{n_\alpha=n} c_\alpha[\alpha]$ is an integral n-cycle, then $|c_\alpha|$ is a constant n. Show in particular that $1/n \cdot c$ is a fundamental cycle and that there are zero or two fundamental cycles on X.

Hint: Show that $|c_\alpha|$ is constant for the n-simplices in a connected component of X.

1.3 The projective plane

So far we haven't seen any examples of a non-orientable pseudomanifold. This is what we shall do in the current subsection where we shall introduce and study the *projective plane*. The projective plane $\mathbf{P}^2$ is obtained from the 2-sphere by identifying antipodal points (see left part of Fig. 6). Note that the lower hemisphere meets all equivalence classes of this relation and it is only the points of the circle in the (x, y)-plane which meet more than one point. Hence we get the projective plane also by taking the lower hemisphere and identifying antipodal points of its points in the (x, y)-plane (see right part of Fig. 6). We can then project the lower hemisphere along the z-axis to get the unit disc and thus the projective plane is also the unit disc with antipodal points of the unit circle identified (see Fig. 7). From this a triangulation can be produced by thinking of this as a square still with antipodal points identified and then divide the square into four triangles. This gives us four triangles G, H, I and J, 6 edges a, b, c, d, e and f and 3 vertices 0, 1 and 2. With the orderings given we have $\partial[G] = [f] - [a] + [e]$, $\partial[H] = [f] - [b] + [c]$, $\partial[I] = [d] - [a] + [c]$, $\partial[J] = [d] - [b] + [e]$, $\partial[a] = [2] - [0]$, $\partial[b] = [2] - [0]$, $\partial[c] = [2] - [1]$, $\partial[d] = [1] - [0]$, $\partial[e] = [2] - [1]$ and $\partial[f] = [1] - [0]$. We can now use this to compute the integral homology of $\mathbf{P}^2$. It is clear that modulo 0-boundaries all points are equivalent and this gives $H_0(\mathbf{P}^2, \mathbf{Z}) = \mathbf{Z}$.

Exercise 10: show that for a triangulated X, the class of a point has infinite order in $H_0(X, \mathbf{Z})$.

Turning to degree 1, it is clear that $c_1 := [b] - [a]$, $c_2 := [f] - [d]$, $c_3 := [e] - [c]$ and $c_4 := [e] + [f] - [b]$ are cycles and it is easy to see that they form a basis for the 1-cycles. We furthermore have $\partial[G] = c_1 + c_4$, $\partial[H] = -c_3 + c_4$, $\partial[I] = c_1 - c_2 - c_3 + c_4$ and $\partial[J] = -c_2 + c_4$. Modulo 1-boundaries we thus have $c_2 \equiv c_3 \equiv c_4$, $c_1 \equiv -c_4$ and then finally $-2c_4 \equiv 0$. This gives us that $H_1(\mathbf{P}^2, \mathbf{Z}) = \mathbf{Z}/2$. Finally, it is easy to see directly that $H_2(\mathbf{P}^2, \mathbf{Z}) = 0$ but we can also

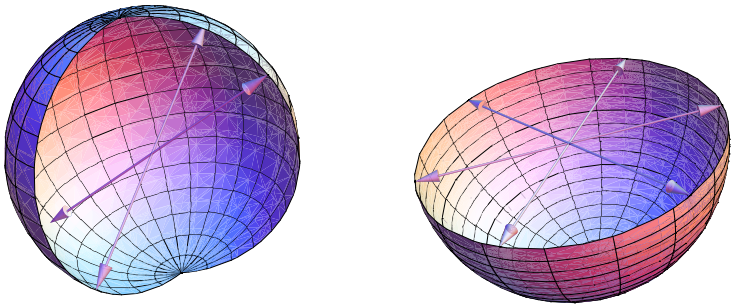

Figure 6: Identifying antipodal points.

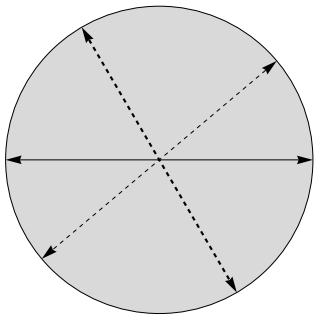

Figure 7: Identifying antipodal points on unit circle.

argue as follows: If we use $\mathbf{Q}$-coefficients we still have $H_1(\mathbf{P}^2, \mathbf{Q}) = 0$ and we get $H_1(\mathbf{P}^2, \mathbf{Q}) = 0$. This means that $\partial_2 \colon C_2(\mathbf{P}^2, \mathbf{Q}) \to C_1(\mathbf{P}^2, \mathbf{Q})$ has a cokernel of dimension 2 and as these vector spaces have rank 4 and 6 respectively we get that ∂_2 is injective.

Exercise 11: Verify the details in this argument.

Exercise 12: Show that $H_2(\mathbf{P}^2, \mathbf{Z}/2) = H_1(\mathbf{P}^2, \mathbf{Z}/2) = H_0(\mathbf{P}^2, \mathbf{Z}/2) = \mathbf{Z}/2$.

1.4 Syzygies

Historically, there is a completely different development which eventually was realised to have to have close connections with homology. This was originally related to the manipulation of polynomials. Hence, for reasons that we won't go into here one was interested in ideals I in polynomial ring $R := \mathbf{k}[x_0, \ldots, x_n]$[9] over a field $\mathbf{k}$ (which at the beginning was assumed to be the complex or possibly the real numbers). Recall that a non-zero[10] polynomial $p \in \mathbf{k}[x_0, \ldots, x_n]$

[9]We start indexing at 0 instead of 1 as it is n that is the appropriate parameter when, as we shall, one is dealing with homogeneous polynomials

[10]The zero polynomial creates something of a problem, sometimes it is better to assume it has no degree, sometimes that it has all possible degrees, we shall often ignore this problem.

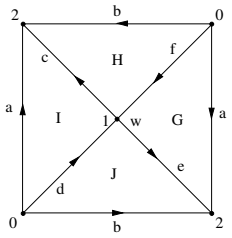

Figure 8: Triangulation of projective plane.

is said to be *homogeneous* of degree k if $p(x_0,\dots,x_n)=\sum_{|\alpha|=k} c_\alpha x^\alpha$, where we use multi-index notation; $\alpha=(\alpha_0,\dots,\alpha_n)\in\mathbf{N}^{n+1}$, $|\alpha|=\alpha_0+\cdots+\alpha_n$ and $x^\alpha=x_0^{\alpha_0}\cdots x_n^{\alpha_n}$. We then use the notation $\deg p=k$ (and by using $\deg p$ we also claim that p is homogeneous). Now, any polynomial p may uniquely be written as a (finite) sum of homogeneous polynomials $p=\sum_k p_k$, where $\deg p_k=k$ and we shall use this notation, p_k, to denote the component of p of degree k. We then say that the ideal I is *homogeneous* if $p\in I \implies \forall k\colon p_k\in I$. Another, more abstract but useful, way of expressing this is to define $I_k:=\{p\in I\mid p=0\vee\deg p=k\}$ and then I is the direct sum[11] $I=\bigoplus_k I_k$. For this reason one also often says that I is a *graded ideal*. This means that R/I is, just like the polynomial ring itself, is a *graded ring*[12], i.e., $R/I=\bigoplus_k (R/I)_k$ and $(R/I)_k\cdot(R/I)_\ell\subseteq(R/I)_{k+\ell}$. Now it turns out that I is finitely generated[13] We may assume that those generators are homogeneous, $I=(f_1,\dots,f_r)$, with $\deg f_i=n_i$ and we may assume that r is minimal. It then turns out that the degrees n_i do not depend on the chosen generators (the generators themselves may, if $\deg f=\deg g$ we may replace them with any basis for the linear space spanned by them also if $\deg f<\deg g$, then g can be replaced by $g+rf$ where $\deg r=\deg g-\deg f$). It was soon realised that in order to understand a graded ideal (with say a chosen minial set of homogeneous generators) it is very useful to know about relations between the generators. Hence a *syzygy*[14] for $(f_1,\dots,f_r)$ is a collection of polymonials $(h_1,\dots,h_r)$ with $\deg h_i+\deg f_i$ independent of i such that $\sum_i h_i f_i=0$. In modern terms the syzygies form the graded components of an R-submodule of R^r. More precisely we should keep track of degrees. Hence for integer e we define $R(e)$ to be the graded R-module[15] with $R(e)_k$, its degree k-part, being equal to R_{k-e} (so that the module generator 1 lies in degree e). If $(f_1,\dots,f_r)$ is a minimal set of homogeneous generators of the ideal I with $e_i:=\deg f_i$, then we define a map $\bigoplus_i R(e_i)\to R$ which maps $(h_1,\dots,h_r)$ to $\sum_i h_i f_i$. It is a homogeneous R-module map (i.e., it preserves degrees) and its image is exactly I. Its kernel K is a graded submodul of $\bigoplus_i R(e_i)$ whose homogeneous elements are exactly the syzygies.

This process can be iterated. Again we have that K is finitely generated and we can find a minimal set of homogeneous elements (which now are r-vectors of polynomials instead of just polynomialis) and then do the same construction again. This leads to the following situation. We get an exact sequence[16] of R-modules

$$\cdots\to\bigoplus_i R(e_i^\ell)\to\bigoplus_i R(e_i^{\ell-1})\to\cdots\to\bigoplus_i R(e_i^2)\to\bigoplus_i R(e_i^1)\to\bigoplus_i R(e_i^0)\to R/I\to 0,$$

called a *resolution* of R/I, where the e_i^k are the degrees of a minimal set of homogeneous generators for the kernel of $\bigoplus_i R(e_i^{k-1})\to\bigoplus_i R(e_i^{k-2})$. The famous syzygy theorem[17] of Hilbert says

[11] For definition see `http://en.wikipedia.org/wiki/Direct_sum`.
[12] For definition see `http://en.wikipedia.org/wiki/Graded_ring`.
[13] Historically this came later and people spent a lot of time proving that specific ideals were finitely generated.
[14] *Syzygy* is a Greek word meaning *pair* or *conjunction*. It is used in a surprisingly large number of contexts. The first mathematical use of it recorded by the Oxford English Dictionary is in an article by Sylvester from 1850.
[15] For definition see `http://en.wikipedia.org/wiki/Graded_module`.
[16] For definition see `http://en.wikipedia.org/wiki/Exact_sequence`.
[17] For definition see `http://en.wikipedia.org/wiki/Syzygy_theorem`.

that the set $\{e_i^\ell\}_i$ is empty when $\ell > n+1$.

Example: i) When $n = 0$ we know that $\mathbf{k}[x_0]$ is a principal ideal domain[18] and hence I (if non-zero) is generated by a single element f_1. The map $R(e_1^1) \to R$ which maps h to hf_1 is injective and its kernel is thus 0.

ii) Suppose that $I = (f_1, f_2)$. As R is a unique factorisation domain[19] we can write $f_i = g_i g$, where g_1 and g_2 are without a common factor. Then $(g_2, -g_1)$ is a syzygy and conversely if $(h_2, -h_1)$ is a syzygy, then $h_2 g_1 g = h_1 g_2 g$ which gives (as R is a domain) $h_2 g_1 = h_1 g_2$ and as g_1 and g_2 have no factor in common we get $g_i | h_i$ so that we can write $h_i = k_i g_i$ but $h_2 g_1 = h_1 g_2$ then gives $k_1 = k_2 = k$, i.e., $(h_2, -h_1) = k(g_2, -g_1)$. This means that the first syzygy module is free with basis $(g_2, -g_1)$ and the process stops. This shows that when $n > 1$ the process does not have to go all the way, Hilbert's result only says that the process *must* stop before n steps.

iii) Consider the ideal $I = (x_0, \dots, x_n)$. For $0 \le i < j \le n$ we have a syzygy $(0, \dots, \overset{i}{x_j}, \dots, \overset{j}{-x_j}, \dots 0)$ and it is not difficult to see that they form a basis for the syzygies. However, as soon as $n > 1$ there are second syzygies. To get the general pattern rather than describing directly the syzygies we instead describe the resolution. Hence, for $0 \le k \le n$ we let S_k be the free graded R-module with basis $\{e_{i_1,\dots,i_k}\}$, where $0 \le i_1 < i_2 < \cdots < i_k \le n$, all of degree k. We define an R-map $d_k \colon S_k \to S_{k-1}$ specified by $d_k(e_{i_1,\dots,i_k}) = \sum_{1 \le r \le k} (-1)^{r+1} x_{i_r} (i_1, \dots, \widehat{i_r}, \dots, i_k)$ (where as usual the hat means that that index is removed).

Exercise 13: Show that this is a minimal resolution of $R/(x_0, \dots, x_n)$.

Hence we get that Hilbert's theorem is sharp if we consider all graded modules.

There is a very nice application of Hilbert's syzygy theorem to the computation that today is known as the *Hilbert function*. It was realised early that the function $k \mapsto H_I(k) := \dim(R/I)_k$ is an important invariant of the ideal I. Of course there is only a finite number of homogeneous monomials of degree k and hence R_k and thus $(R/I)_k$ is finite dimensional.

Exercise 14: Show that $\dim R_k = \binom{n+k}{n}$.

We now apply Hilbert's syzygy theorem to get a minimal resolution

$$0 \to S_{n+1} \to S_n \to \cdots \to S_1 \to S_0 \to R/I \to 0,$$

where $S_i = \bigoplus_j R(e_j^i)$. If we look at a specific degree this gives an exact sequence of finite dimensional vector spaces

$$0 \to (S_{n+1})_k \to (S_n)_k \to \cdots \to (S_1)_k \to (S_0)_k \to (R/I)_k \to 0$$

and as a consequence we get $H_I(k) = \sum_i (-1)^i \dim(S_i)_k$. Now, $\dim R(e)_k = \dim R_{k-e} = \binom{n+k-e}{n}$ and hence

$$H_I(k) = \sum_{i,j} (-1)^i \binom{n+k-e_j^i}{n}.$$

This has an interesting qualitative consequence.For $\ell \ge 0$ we have that $m \mapsto \binom{m}{\ell}$ is the polynomial function $m \mapsto m(m-1)\cdots(m-\ell+1)/\ell!$. Note that while this polynomial makes sense for all m it does not equal $\binom{m}{\ell}$ for general $m < 0$ (to fit in with the above exercise we should have $\binom{m}{\ell} = 0$ when $m < \ell$). So our general conclusion is that $M_I(k)$ is a polynomial function for $k >> 0$.

The fact that the Hilbert function is a polynomial for large k can be proved in different ways (arguably simpler). However, this view point has turned out to be extremely important for the more detailed study of the Hilbert function.

It may seem that Hilbert's syzygy theorem is conceptually different from homology in that it deals with exact sequences while one of the major point of homology is that it often is non-trivial which means exactly that the sequence of chains and boundary maps is not exact. However

[18] For definition see `http://en.wikipedia.org/wiki/Principal_ideal_domain`.

[19] For definition see `http://en.wikipedia.org/wiki/Unique_factorization_domain`.

when people tried to understand why the e^i_j are independent of the chosen bases (and also to generalise it to situations where minimal resolutions may not exist such as for non-graded ideals) it was realised that one should look at the sequence (for the tensor product see [20])

$$0 \to S_{n+1} \bigotimes_R \mathbf{k} \to S_n \bigotimes_R \mathbf{k} \to \cdots \to S_1 \bigotimes_R \mathbf{k} \to S_0 \bigotimes_R \mathbf{k} \to 0,$$

where $\mathbf{k}$ is considered as an R-module by regarding it as the quotient $R/(x_0, \ldots, x_n)$ which is no longer exact and then consider its homology. The independence of the e^i_j then follows from an independence result of this homology of the chosen resolution. This is analogous to the claims that have been repeated made about the independence of the homology of a topological space of for instance the choice of a triangulation.

1.5 Group homology

One of the first striking realisation that algebraic topology (homology etc) and homological algebra (minimal resolutions etc) were very closely related was the realisation that the homology of Eilenberg-MacLane spaces had a purely algebraic description. The situation is the following:

- Recall that a (Hausdorff say) space X is contractible[21] if it is non-empty and there is a continuous map $F\colon X \times [0,1] \to X$ such that $F(-,0)\colon X \to X$ is the identity map and $F(-,1)\colon X \to X$ is constant.
- A group G acting continuously on X (i.e., $G \times X \to X$ is a continuous map) acts *action!free*freely on X if for all $x \in X$ $gx = x$ implies that $g = e$.
- An action is *proper* if each $x \in X$ has a neighbourhood $x \in U \subseteq X$ such that $gx \in U \implies g = e$. In particular the action is free.
- The relevant result (proved by Eilenberg[22] and Mac Lane[23]) is then that if one considers the quotient space X/G of a proper action on a contractible space X (which is a space as nice as X because the action is proper), then its homology $H_k(X/G, \mathbf{Z})$ only depends on the group G and was eventually also denoted $H_k(G, \mathbf{Z})$, the *group homology* of G.

Example: i) The action of $\mathbf{Z}$ and $\mathbf{R}$ given by $n \cdot r = r + n$ is proper and $\mathbf{R}$ is contracible by the map $(r,t) \mapsto (1-t)r$. The map $\mathbf{R} \to S^1$ given by $r \mapsto e^{2\pi i r}$ identifies $\mathbf{R}/\mathbf{Z}$ with the circle. We have already computed the homology of the circle so that we get $H_0(\mathbf{Z}, \mathbf{Z}) = \mathbf{Z}$ and $H_1(\mathbf{Z}, \mathbf{Z}) = \mathbf{Z}$ and all other homology groups are equal to zero.

ii) A more complicated example is concerned with the group $\mathbf{Z}/2$. We define the infinite dimensional sphere S^∞ as $S^\infty := \{(x_i) \in \mathbf{R}^{\mathbf{N}} \mid x_i = 0 \text{ for } i >> 0, \sum_i x_i^2 = 1\}$ (where of course the index from which the x_i vanish is allowed to depend on the vector). It is the union of the finite dimensional spheres S^n can be identified with the subset of S^∞ for which $x_i = 0$ when $i > n$. We have $\mathbf{Z}/2$ acting on S^∞ by $\overline{1} \cdot (x_i) = (-x_i)$, i.e., it maps a point to its antipodal point. This action is proper and the quotient space can, by analagy with how we defined the projective plane, be called the infinite dimensional projective space $\mathbf{P}^\infty$. We shall find a triangulation of $\mathbf{P}^\infty$ and compute its homology thus showing that

$$H_k(\mathbf{Z}/2, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{when } k = 0, \\ 0 & \text{when } k \text{ is odd}, \\ \mathbf{Z}/2 & \text{when } k > 0 \text{ is even}. \end{cases}$$

iii) It is possible to go the other way around: We may have a topological space Y which is of the form X/G and then (provided that we have an algebraic description of $H_k(G, \mathbf{Z})$, as we do)

[20] For definition see `http://en.wikipedia.org/wiki/Tensor_product`.
[21] For definition see `http://en.wikipedia.org/wiki/Contractible`.
[22] Samuel Eilenberg, 1913–1998
[23] Saunders Mac Lane, 1909–2005

compute its homology by computing the group homology of G. Such a space is denoted $K(G,1)$ or BG (as it to a large enough extent only depends on G).

iv) Examples of $K(G,1)$'s are given by closed orientable (say) surfaces[24] which are classified by their genus $g \geq 0$. Except for $g = 0$ (which is the sphere S^2) they are $K(\pi_g, 1)$'s where the group π_g is a very interesting interesting group. The case of $g = 1$ is a little bit special as the surface is just $S^1 \times S^1 = \mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z} = \mathbf{R}^2/\mathbf{Z}^2$, which tells us that $\pi_1 = \mathbf{Z}^2$. For larger g π_g is non-commutative with very interesting properties. (To be truthful this is probably not a good example of an algebraic computation being easier than the topological as it is very easy to do the calculation of the homology of a closed surface.)

v) Another example of a $K(G,1)$ is a wedge[25] of circles, a collection of circles with all elements $1 \in S^1$ identified. The group G in question is the free group[26] on generators one for each circle.

vi) There are (surprisingly) many naturally occurring examples of $K(G,1)$'s. Often the groups are natural examples of group (such as $\mathrm{SL}_n(\mathbf{Z})$, the group of integer matrices with determinant 1) but are fairly complicated. Hence, their group homology is usually studied by combining algebraic and topological techniques.

One method to give an algebraic description of group homology is as follows: One can reduce (using the indepence result already mentioned) to the case when X has a triangulation and G permutes the simplices. One may also assume[27] that only the identity element of G maps a simplex to itself. This means that the k-chains $C_k(X, \mathbf{Z})$ is a module over the group ring[28] $\mathbf{Z}[G]$ and as such it is free module. Indeed, by picking representatives for the orbits of the G-action on the k-simplices we get a basis. Furthermore, the differential $\partial_k \colon C_k(X, \mathbf{Z}) \to C_{k-1}(X, \mathbf{Z})$ is a $\mathbf{Z}[G]$-module map. Now, the fact that X is contractible implies that its homology is trivial, i.e., $H_k(X, \mathbf{Z}) = 0$ for $k > 0$ and $H_0(X, \mathbf{Z}) = \mathbf{Z}$. This means that we have an exact sequence

$$\cdots \to C_k(X, \mathbf{Z}) \to C_{k-1}(X, \mathbf{Z}) \to \cdots \to C_1(X, \mathbf{Z}) \to C_0(X, \mathbf{Z}) \to \mathbf{Z} \to 0$$

and if we let G act trivially on $\mathbf{Z}$ then it is a sequence of $\mathbf{Z}[G]$-modules. This is then interpreted as saying that $\cdots \to C_k(X, \mathbf{Z}) \to \cdots \to C_0(X, \mathbf{Z})$ is a free (as the modules involved are free) resolution of the module $\mathbf{Z}$. This is completely analogous to the case of Hilbert's syszygy theorem where we constructed a (minimal, graded) free resolution of R/I. The next step is to use the fact that X/G acquires an induced triangulation from that of X, the simplices are the images of the simplices of X and two simplices of X map to the same simplex of X/G if one can be transformed into the other by an element of G, i.e., the simplices of X/G correspond to G-orbits of the simplices of X. This fits very well in with the tensor product operation: If $F = \mathbf{Z}[G]e$ is the free module with basis e, then $F \bigotimes_{\mathbf{Z}[G]} \mathbf{Z}$ (where G acts trivially on $\mathbf{Z}$) is a free $\mathbf{Z}$-module with basis $e \otimes 1$ and more generally if F is a free $\mathbf{Z}[G]$-module with basis e_α, then $F \bigotimes_{\mathbf{Z}[G]} \mathbf{Z}$ is a free $\mathbf{Z}$-module with basis $e_\alpha \otimes 1$. From this it follows that $C_k(X, \mathbf{Z}) \bigotimes_{\mathbf{Z}[G]} \mathbf{Z}$ is the free module with basis in bijection with the G-orbits on the k-simplices of X, i.e., $C_k(X, \mathbf{Z}) \bigotimes_{\mathbf{Z}[G]} \mathbf{Z}$ can be identified with $C_k(X/G, \mathbf{Z})$. Under this identification the induced map $\partial_k \otimes 1 \colon C_k(X, \mathbf{Z}) \bigotimes_{\mathbf{Z}[G]} \mathbf{Z} \to C_{k-1}(X, \mathbf{Z}) \bigotimes_{\mathbf{Z}[G]} \mathbf{Z}$ can be identified with $\partial_k \colon C_k(X/G, \mathbf{Z}) \to C_{k-1}(X/G, \mathbf{Z})$ and thus the homology of the complex $\cdots \to C_k(X, \mathbf{Z}) \bigotimes_{\mathbf{Z}[G]} \mathbf{Z} \to \cdots \to C_0(X, \mathbf{Z}) \bigotimes_{\mathbf{Z}[G]} \mathbf{Z}$ can be identified with the homology of X/G. It is then a fundamental theorem of homological algebra that this homology does not depend one the chosen resolution but only on the module (in this case $\mathbf{Z}$ with trivial action). (This is the algebraic version of the Eilenberg-MacLane result that the homology of X/G for a proper action of G on the contractible space X.) In fact for this reason, if one has a ring S, a module[29] M together with a free resolution $F_\cdot \to M$ and a module[30] N, then the i'th homology of $F_\cdot \bigotimes_S N$

[24] For definition see `http://en.wikipedia.org/wiki/Surface\#Closed_surfaces`.
[25] For definition see `http://en.wikipedia.org/wiki/Wedge_sum`.
[26] For definition see `http://en.wikipedia.org/wiki/Free_group`.
[27] This is actually automatic, using the Brouwer fixed point theorem.
[28] For definition see `http://en.wikipedia.org/wiki/Group_ring`.
[29] In general case this should be a right module. For commutative rings or group algebras this does not matter.
[30] Left module!

is denoted $\mathrm{Tor}_i^S(M,N)$. Hence our argument gives the formula $H_k(X/G,\mathbf{Z}) = \mathrm{Tor}_k^{\mathbf{Z}[G]}(\mathbf{Z},\mathbf{Z})$. Similarly, in the Hilbert case we have $\mathrm{Tor}_i^R(R/I,\mathbf{k}) = \bigoplus_j \mathbf{k}(e_j^i)$ showing that the degrees of the minimal generators are in fact independent of the resolution.

Example: The triangulation[31] of S^∞ alluded to above gives the following complex. To make it readable we denote by σ the non-identity element of $\mathbf{Z}/2$ and by 1 the identity element. With that notation the group algebra $\mathbf{Z}[\sigma]$ has 1 (with acts as its identity element) and σ with the only non-trivial part of its multiplication table given by $\sigma^2 = 1$ (so that it can be thought of as $\mathbf{Z}[x]/(x^2-1)$). The complex then is

$$\cdots \to \mathbf{Z}[\sigma] \xrightarrow{\sigma+1} \mathbf{Z}[\sigma] \xrightarrow{\sigma-1} \to \cdots \xrightarrow{\sigma-1} \mathbf{Z}[\sigma] \to \mathbf{Z} \to 0.$$

Note that $(\sigma+1)(\sigma-1) = \sigma^2 - 1 = 0$ so that the composite of two adjacent maps is indeed 0. That it is exact is a simple computation. Tensoring with $\mathbf{Z}$ turns each $\mathbf{Z}[\sigma]$ into a $\mathbf{Z}$ and as σ acts as 1 on $\mathbf{Z}$, $\sigma - 1$ acts as $1-1=0$ and $\sigma+1$ as $1+1=2$. This means that the tensored resolution becomes

$$\cdots \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \to \cdots \xrightarrow{0} \mathbf{Z}$$

and the homology of this complex is clearly $\mathbf{Z}$ in degree 0, $\mathbf{Z}/2$ in all other even degrees and 0 in odd degrees.

Even though the definition of group homology was discovered first when one knew that the homology of an Eilenberg-MacLane space only depended on the group, low degree homology had been independently discovered previously.

- We of course have $H_0(G,\mathbf{Z}) = \mathbf{Z}$.
- We have that $H_1(G,\mathbf{Z}) = G/[G,G]$, where $[G,G]$ is the subgroup generated by commutators $[g,h] := ghg^{-1}h^{-1}$ (which is a normal subgroup). This is a commutative group, in fact it is the maximal commutative quotient of G and as such is of great interest in group theory.
- $H_2(G,\mathbf{Z})$ is the so called *Schur*[32] *multiplier*. Knowing it allows one to understand *central extensions* of G; surjective maps $H \to G$ such that the kernel lies in the center[33] of H.

[31] Technically, as we shall see, it is not quite a triangulation but a generalisation called a cell decomposition.
[32] Issai Schur, 1875 - 1941
[33] For definition see `http://en.wikipedia.org/wiki/Group_center`.