Matematiska Institutionen
KTH

Exam to the course Discrete Mathematics, SF2736, December 19, 2011, 14.00–19.00.

Observe:

1. You are not allowed to use anything else than pencils, rubber, rulers and papers at this exam.
2. To get the maximum number of points on a problem it is not sufficient to just give an answer, you must also provide explanations.
3. Bonus points from the homeworks will be added to the sum of the points on part I.
4. Grade limits: 13-14 points will give an Fx; 15-17 points will give an E; 18-21 points will give a D; 22-27 points will give a C; 28-31 points will give a B; 32-36 points will give an A.

Part I

1. (3p) Draw a bipartite graph with 8 vertices and 12 edges that has an Euler circuit but no Hamiltonian cycle and draw another bipartite graph, also with 8 vertices and 12 edges, that has an Hamiltonian cycle but no Euler circuit.
2. (3p) Use the technique with generating functions to find all sequences $a_0, a_1, a_2, \ldots$ that satisfy the recursion
$$a_{n+2} = 7a_{n+1} - 10a_n \qquad \text{for} \qquad n = 0, 1, 2, \ldots .$$
3. (3p) John will, as a Christmas present, get a package with ten bolls. They are colored either green, red, yellow or blue. How many possible distinct packages are there, if a package must contain at least one ball of each color? The solution shall, besides explanations, also contain an answer, to the question, given as an integer.
4. (3p) Find the least positive remainder when 37^{121} is divided by 42.
5. (3p) Find the smallest subgroup H of the group $G = (Z_{30}, +)$ with the property that both 3 and 8 belongs to the same coset of H in G.

Part II

6. (3p) A tree is a connected graph without any cycles. A forest is a graph that consists of trees. Find the least number of trees, as well as the largest number of trees, in a forest with 100 vertices if at least 27 of the vertices has degree 3.

7. (4p) Find the number of equivalence relations on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ such that 1 and 2 are not equivalent, 2 and 3 are not equivalent, 3 and 4 are not equivalent. The solution shall, besides explanations, also contain an answer, to the question, given as an integer.

8. Let $\mathcal{S}_n$ denote the group that consists of all permutations of the elements in the set $\{1, 2, 3, \ldots, n\}$.

   (a) (1p) Find all cyclic subgroups of order 4 in $\mathcal{S}_4$.

   (b) (1p) Find the number of cyclic groups of order 4 in $\mathcal{S}_n$, for $n \geq 4$.

   (c) (2p) Give a formula for the number of cyclic groups of order p in $\mathcal{S}_n$, for every prime number $p \leq n$.

Part III

9. For an 1-error correcting code C of length n we will below denote the set of words that C cannot correct by $D(C)$, i.e., $D(C)$ denotes the set of words at distance at least two from each of the words in C.

   (a) (1p) The 1-error correcting code C is linear and the matrix $\mathbf{H}$ below is a parity check matrix for C:

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   Show that the set of words $D(C)$ is a 1-error correcting code that is not linear.

   (b) (2p) Show that, for every linear 1-error correcting code C of length n and size $|C| = 2^{n-m}$ with a parity check matrix $\mathbf{H}$ with m rows and $n = 2^m - 2$ columns, the set of words $D(C)$ is a 1-error correcting code that is not linear.

   (c) (2p) Can the statement above be true if the matrix $\mathbf{H}$ has m rows and $n = 2^m - 3$ columns? Always, under certain conditions, or never?

10. Let G be a finite group and a and b two elements in G such that $ab = ba$. Let $\sigma(g)$ denote the order of an element g in G.

    (a) (2p) Show that if $\gcd(\sigma(a), \sigma(b)) = 1$ then $\sigma(ab) = \operatorname{lcm}(\sigma(a), \sigma(b))$.

    (b) (3p) Can $\sigma(ab) = \operatorname{lcm}(\sigma(a), \sigma(b))$ if $\gcd(\sigma(a), \sigma(b)) \neq 1$? Always, under certain conditions, or never? (A correct guess will give 1p.)