Matematiska Institutionen
KTH

Solutions to homework number 2 to SF2736, fall 2011.

1. (0.2p) Let $M = \{1,2,3,4,5,6,7\}$. Describe all equivalence relations $\mathcal{R}$ on M such that

$$\{(1,5),(1,4),(2,3),(3,6)\} \subseteq \mathcal{R}.$$

Solution: Every equivalence relation is uniquely described by its equivalence classes. By transitivity and symmetry we get that the elements 1, 5 and 4 must belong to the same equivalence class, and that 2, 3 and 6 must belong to the same equivalence class. So one possibility to partition M into equivalence classes, with the given property on the equivalence relation is

$$M = C_1 \cup C_2 \cup C_7,$$

where $C_1 = \{1,5,4\}$, $C_2 = \{2,3,6\}$ and $C_7 = \{7\}$. Then, if some element in C_1 is related to some element in C_2, then all elements in these equivalence classes are related to each other and we get the partition

$$M = \{1,2,3,4,5,6\} \cup \{7\}.$$

Similarly we may derive another three partitions of M:

$$M = \{1,5,4,7\} \cup \{2,3,6\}$$

and

$$M = \{1,5,4\} \cup \{2,3,6,7\}$$

and

$$M = \{1,2,3,4,5,6,7\}.$$

These five partitions of M describe the feasible equivalence relations.

2. (0.2p) Let N denote the set $\{0,1,2,3,\ldots\}$. Find and give an explicit description of a bijection from $N \times N$ to N.

Solution: We enumerate the elements in $N \times N$ in the following order

$$(0,0)$$

as the first element. Next the following two:

$$(1,0) \qquad (0,1)$$

The next three elements will be

$$(2,0) \qquad (1,1) \qquad (0,1)$$

And the next four will be

$$(3,0) \qquad (2,1) \qquad (1,2) \qquad (0,3)$$

And so on. We observe that the sum $n+m$ of the components in the pair (n,m) is equal to the number of pairs in the preceding row. Thus we have enumerated in total

$$1+2+\ldots+(n+m) = \frac{(n+m)(n+m+1)}{2}$$

pairs when we continue the enumeration with the pairs from the row of (n,m).

The bijective map $f : N \times N \to N$ that describes the enumeration starts with

$$f(0,0) = 0.$$

Hence, for $k = 1, 2, \ldots$,

$$f(k,0) = \frac{k(k+1)}{2},$$

and continuing the enumeration we get for $r = 0, 1, \ldots, k$,

$$f(k-r,r) = \frac{k(k+1)}{2} + r$$

Now, with $k = n+r$ and $m = r$, we arrive with the explicit formula

$$f(n,m) = \frac{(n+m)(n+m+1)}{2} + m.$$

3. (0.3p) Is the set of functions from the set of positive integers Z^+ to the set $\{0,1\}$ an infinite countable set? Explain your answer with great care!

Solution: Every map f from $Z^+ = \{1,2,3,\ldots\}$ to $\{0,1\}$ corresponds to an infinite binary sequence

$$f = (f(1), f(2), f(3), \ldots)$$

Assume there is an enumeration of all such functions:

$$f_1, f_2, f_3, \ldots\ .$$

Every function is then included in this enumeration. However, the function f defined by

$$f(i) \neq f_i(i),$$

for $i = 1, 2, \ldots$, is not contained in this enumeration. To see this, assume that $f = f_j$ for some j. As $f(j) \neq f_j(j)$, this cannot be the case.

The assumption that the enumeration of functions above contains all functions is thus wrong. We may consequently conclude that no enumeration of functions from Z^+ to $\{0,1\}$ can contain all functions.

4. (0.3p) Is the set of bijections from Z^+ to Z^+ an infinite countable set? Explain your answer with great care!

Solution: Let $\mathcal{B}$ denote the set of all bijections from Z^+ to Z^+, and let $\mathcal{S}$ denote the family of all infinite subsets of Z^+. We shall define an injective map $I : \mathcal{S} \to \mathcal{B}$. By the diagonalization method of Cantor it is clear that $\mathcal{S}$ is not countable infinite. Then the subset $I(\mathcal{S})$ of $\mathcal{B}$ is not countable infinite, and, as a countable infinite set cannot contain a not countable infinite set, it follows that $\mathcal{B}$ cannot be countable infinite.

Now to the definition of the function I. To any subset $S = \{i_1, i_2, i_3, \ldots\}$ of Z^+ we associate a bijection f from Z^+ into itself, $f = I(S)$, defined by

$$f(i) = \begin{cases} i & \text{if } i \notin S \\ i_{2k} & \text{if } i = i_{2k-1} \\ i_{2k-1} & \text{if } i = i_{2k} \end{cases}$$

where k takes the values $1, 2, 3, \ldots$. Clearly, from the definition of f follows that $f = I(S)$ is a bijective map for every infinite subset S of Z^+. As furthermore $I(S)(j) = j$ if and only if $j \notin S$, we may conclude that the map I is injective. The problem is solved.