Matematiska Institutionen
KTH

Solutions to homework number 3 to SF2736, fall 2011.

1. (0.2p) In a class consisting of 12 boys and 13 girls a committee consisting of five children has to be chosen. In how many ways can this be done if the boy B refuses to attend if the boy P attends the committee, and the girl F must attend if the girl G attends.

   Solution: If there were no restrictions then you could choose any five children among the 25, and then the number of possible committees would be:
$$|\mathcal{X}| = \binom{25}{5}.$$
   The forbidden committees are those where both B and P attend the committee, which be the committees in the set $\mathcal{A}$ of committees and the committees when the girl G attends and not the girl F attends, which a set of committees that will be denoted by $\mathcal{B}$.

   The answer is given by
$$|\mathcal{X}| - |\mathcal{A} \cup \mathcal{B}|.$$
   If P and B are chosen to the committee then it remains to choose another three children among the remaining 23 so
$$|\mathcal{A}| = \binom{23}{3}.$$
   If G is chosen but not F, then it remains 23 children among which we have to choose four, so
$$|\mathcal{B}| = \binom{23}{4}.$$
   The number of committees that belong both to $\mathcal{A}$ and $\mathcal{B}$ is
$$\binom{21}{2},$$
   as the two boys must be chosen, as well as the girl G but not the girl F, so remains 21 children from which another 2 must be chosen.

Inclusion exclusion now gives

ANSWER:

$$\binom{25}{5} - \binom{23}{3} - \binom{23}{4} + \binom{21}{2}$$

2. (0.2p) Find the number of 4-tuples (x_1, x_2, x_3, x_4) of non negative integers that satisfy the relation

$$5 \leq x_1 + x_2 + x_3 + x_4 \leq 15$$

The answer must be given as an integer.

Solution: The answer is given by the difference between the number of solutions to the following two inequalities

$$x_1 + x_2 + x_3 + x_4 \leq 15 \qquad \text{and} \qquad x_1 + x_2 + x_3 + x_4 \leq 5.$$

We introduce a so called slack variable y, which takes care of the difference between the left side of the inequality and the right side. So the number of solutions to the first of the above two inequalities is equal to the number of solutions of

$$x_1 + x_2 + x_3 + x_4 + y = 15,$$

where all variables are non negative, and similarly for the other inequality. From the know general formula for the number of solutions to this kind of systems we get

ANSWER:

$$\binom{15+4}{4} - \binom{4+4}{4} = \frac{19 \cdot 18 \cdot 17 \cdot 16}{4 \cdot 3 \cdot 2} - \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 3806.$$

3. (0.2p) Find the coefficient of $x^2 y^3 z$ in the polynomial

$$(x + 4y - az + 3)^{17}$$

Solution: The generalization of the binomial theorem to the case of more than two entries is, for four entries x, y, z and w:

$$(x + y + z + w)^n = \sum \binom{n}{k, l, r, t} x^k y^l z^r w^t,$$

where the summation is for all 4-tuples (k, l, r, t) such that

$$k + l + r + t = n \qquad \text{and} \qquad k \geq 0, l \geq 0, r \geq 0, t \geq 0.$$

So if we let $x = x$, $y = 4y$, $z = -z$ and $w = 3$ in the formula above, we get that the coefficient of x^2y^3z is achieved when $k = 2$, $l = 3$, $r = 1$ and $t = 17 - k - l - r = 11$. So by the formula we get

ANSWER:

$$-\binom{17}{2, 3, 1, 11} 4^3 a 3^{11}.$$

4. (0.2p) Find the number of words of length 12 consisting of four a's, four b's and four c's with the property that no two a's are adjacent.

   Solution: We first form words consisting of four b's and four c's. As the length of any such word is eight and we must choose four positions in the word to the b's and the remaining four positions to the c's, the number of such words is equal to

   $$\binom{8}{4}.$$

   Next we put in the a's. No two a's can be put in in the same of the nine spaces between the letters b and c. In the picture below we give a picture of the situation. The b's and c's are denoted by x:

   x x x x x x x x.

   So there are

   $$\binom{9}{4}$$

   possibilities to put in the a's, such that no two a's will be adjacent.

   ANSWER:

   $$\binom{8}{4} \cdot \binom{9}{4}.$$

5. (0.2p) Find the number of words of length 8 in the letters a, b and c with the property that every word contains at least one a, at least one b and at least one c.

Solution: Let X denote the set of words without the letter x. Then

$|A| = 2^8 = |B| = |C|$, $|A \cap B| = 1 = |A \cap C| = |B \cap C|$, $|A \cap B \cap C| = 0$.

So by the principle of inclusion and exclusion we get
ANSWER:

$$3^8 - 3 \cdot 2^8 + 3 \cdot 1 - 0.$$