Matematiska Institutionen
KTH

Solutions to homework number 5 to SF2736, fall 2011.

1. (0.2p) Use the technique with generating functions to find explicit expressions for the numbers a_n, if this sequence of numbers satisfies

$$a_n = a_{n-1} + 12a_{n-2}, \qquad n = 2, 3, 4, ...$$

and $a_0 = 0$, $a_1 = 7$.

Solution: Let

$$A(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$$

Multiplying the given equality with t^n, for $n = 2, 3, \dots$ and sum we obtain the equality

$$A(t) - 7t = tA(t) + 12t^2 A(t),$$

so

$$A(t) = \frac{7t}{1 - t - 12t^2}.$$

Partial fractions

$$A(t) = \frac{7t}{(1-4t)(1+3t)} = \frac{1}{1-4t} - \frac{1}{1+3t}$$

and sum of a geometrical series

$$A(t) = \sum_{n=0}^{\infty}(4t)^n - \sum_{n=0}^{\infty}(-3t)^n = \sum_{n=0}^{\infty}(4^n - (-3)^n)t^n.$$

Thus

ANSWER: $a_n = 4^n - (-3)^n$.

2. (0.2p) Find the number of ways to color the edges of a tetrahedron in five colors.

Solution: The corners are enumerated 1, 2, 3, 4, and the edge between 1 and 2 by a, 1 and 3 by b, 1 and 4 by c, 2 and 3 by d, 2 and 4 by e and the edge between 3 and 4 by f.

The automorphism group H of the tetrahedron is of size 12, as the corner 1 can be rotated to any of the four corners x and then we can rotate the other three corners distinct from x. We then get the group

$$H = \{\text{id.}, (1)(2\ 3\ 4), (1)(2\ 4\ 3), (1\ 2)(3\ 4), (1\ 2\ 3)(4), (1\ 2\ 4)(3),$$

$$(1\ 3)(2\ 4), (1\ 3\ 4)(2), (1\ 3\ 2)(4), (1\ 4)(2\ 3), (1\ 4\ 3)(2), (1\ 4\ 2)(3)\}.$$

These permutations induces permutations on the set of edges, which constitutes the group G. We will apply the lemma of Burnside and hence we organize the following table (the elements of G are ordered following the enumeration above for H):

$\varphi \in G$	$\lvert\text{Fix}(\varphi)\rvert$
$(a)(b)(c)(d)(e)(f)$	5^6
$(a\ b\ c)(d\ f\ e)$	5^2
$(a\ c\ b)(d\ e\ f)$	5^2
$(a)(b\ c)(d\ e)(f)$	5^4
$(a\ d\ b)(c\ e\ f)$	5^2
$(a\ e\ c)(b\ d\ f)$	5^2
$(b)(e)(b\ f)(c\ e)$	5^2
$(a\ d\ e)(b\ f\ c)$	5^2
$(a\ b\ d)(c\ f\ e)$	5^2
$(c)(d)(a\ f)(b\ e)$	5^4
$(a\ e\ d)(b\ c\ f)$	5^2
$(a\ c\ e)(b\ f\ d)$	5^2

Now, applying the lemma of Burnside, we get

ANSWER:

$$\frac{1}{12}(5^6 + 3 \cdot 5^4 + 8 \cdot 5^2) = 1475.$$

3. (0.2p) Find a linear 1-error correcting code C, containing as many words as possible, among them the word 1110011010, and such the word 0001111101 cannot be corrected (or does not belong to C).

Solution: We give the parity check matrix $\mathbf{H}$ for the code. The length of the words is 10, so the number of columns of $\mathbf{H}$ must be 10. These columns must be distinct and no zero column appear, so there must be at least four rows (with three rows you get at most seven distinct non zero columns). Now, with some trial and error, we try to construct a 4×10-matrix that multiplied with the transpose of the first word will be the zero column, and by multiplication with the transpose of the second word shall be a column that does not appear in the matrix. So after some trials we get

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

4. (0.4p) Find a binary matrix $\mathbf{H}$ such that the set of words

$$C = \{\bar{x} = (x_1, x_2, \ldots, x_n) \mid \mathbf{H}\bar{x}^T = \bar{0}^T\}$$

constitutes a 2-error correcting code of length $n = 10$ and with as many words as possible.

Solution: The code C is linear, so minimum distance 5 is equivalent to say that the minimum weight of the words shall be 5. These means that the sum of four or fewer columns never can be the zero column.

The rows of $\mathbf{H}$ can be chosen to be linear independent, and elementary row operations do not change the null space of a matrix, so we may assume, without loss of any generality, that the first columns consists of just one one and then zeros.

As the sum of four or less columns shalt not be the zero column, we then cannot have further columns with $d = 3$ or less ones, as a such column combined with a suitable choice of d of the first columns will be the zero column.

Again a trial and error search gives that we must have at least seven rows, and in fact the following matrix will do:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$