Institutionen för matematik
**KTH**
Chaotic Dynamical Systems, SF2720, Fall 2012

## Homework assignment 3

The exercises are due on November 23, 2012

(1) Let $\omega \in \Omega_2^R$ (see definition on page 214). Define $W^s(\omega)$ to be the set of sequences $\omega' \in \Omega_2^R$ such that $d(\sigma^n(\omega), \sigma^n(\omega')) \to 0$ as $n \to \infty$. Identify all the sequences in $W^s(\omega)$.

(2) Let $g_c(x) = x^2 + c$. Prove that if $c < 1/4$, there is a unique $a > 1$ such that $g_c(x)$ is (topologically) conjugate to $f_a(x) = ax(1-x)$ via a map of the form $h(x) = \alpha x + \beta$.

(3) A point $p$ is a *non-wandering* point for the map $f$, if for any open set $U \ni p$, there exists $x \in U$ and $n > 0$ such that $f^n(x) \in U$ (note that we do not require that $p$ itself return to $U$). Let $NW(f)$ denote the set of all non-wandering points for $f$.
a) Prove that $NW(f)$ is a closed set.
b) Let $f$ and $\Lambda$ be as in Proposition 7.4.4 (on page 223). Show that $NW(f) = \Lambda$.

(4) A point $p$ is *recurrent* for $f$ if for any open set $U \ni p$ there exists $n > 0$ such that $f^n(p) \in U$.
a) Let $f$ be as in Proposition 7.4.4. Give an example of a non-periodic recurrent point for $f$.
b) Give an example of a non-wandering point for $f$ (with $f$ as in a) ) which is not recurrent.

(5) Let $f : I \to I$ be a smooth map. The *Lyapunov exponent* $L(x)$ is defined as

$$L(x) = \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{d}{dx} f^n(x) \right|,$$

provided the limit exists.

a) Given $x_0 \in I$, let $x_k = f^k(x)$, $k \geq 1$. Assume that $L(x_0)$ exists. Show that

$$L(x_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)| \, .$$

b) Let

$$f(x) = \begin{cases} 2x, & x \in [0, 1/2] \\ 2(1-x), & x \in [1/2, 1] \end{cases} .$$

If $x_0 \in [0, 1]$ is such that $x_k \neq 1/2$ for all $k \geq 0$, show that $L(x_0) = \ln 2$.

c) Let $f(x) = (3/2)x(1-x)$. Compute $L(x_0)$ for $x_0 \in (0, 1)$.

(6) Let

$$f(x) = \begin{cases} 3x, & x \leq 1/2 \\ 3(1-x), & x > 1/2 \end{cases} .$$

a) Show that $f^n(x) \to -\infty$ as $n \to \infty$ for all $x \notin [0, 1]$.

b) Let $\Lambda = \{x : f^n(x) \in [0, 1] \text{ for all } k \geq 0\}$. What kind of set is $\Lambda$?

c) Show that the restriction of $f$ to $\Lambda$ (i.e., $f|_\Lambda$) is conjugate to the 2-shift $\sigma^R$.