Institutionen för matematik
**KTH**
Chaotic Dynamical Systems, Fall 2012

## Homework assignment 4

The exercises are due on December 14, 2012

1 a). Show that a volume-preserving map does not have an attracting fixed point.
b). Let $X$ be a closed domain of finite volume in $\mathbb{R}^n$, and $f : X \to X$ a continuous invertible map with an attracting fixed point. Show that the set of recurrent points for $f$ is not dense in $X$.

2. Let $F^n(p) = p$. A point $p$ is a *sink* (or *attracting periodic point*) if all of the eigenvalues of $DF^n(p)$ are less than one in absolute value; $p$ is a *source* (or *repelling periodic point*) if all of the eigenvalues of $DF^n(p)$ are larger than one in absolute value; $p$ is a *saddle* if some eigenvalues are larger, and some smaller than one in absolute value.

   Consider the torus parametrized by $x$, $y$ in the square $0 \leq x, y \leq 2\pi$ with sides identified. For a small $\varepsilon > 0$, define
$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \varepsilon \sin x \\ y + \varepsilon \sin y \cos x \end{pmatrix}.$$
   a). Find all fixed points of $F$ on the torus.
   b). Classify each of these points as sources, sinks or saddles.
   c). If a point is a saddle, identify and sketch the corresponding stable and unstable invariant manifolds.
   d). Use those to sketch other trajectories of $F$.

3. Write a functional of 3 variables whose critical points correspond to 3-periodic orbits of a given convex billiard. (Read Sec. 6.4.2.)

   The following problem is not mandatory. Following the discussion in Sec. 14.1.1 and 14.1.2, use the above functional to prove that every convex billiard has at least one 3-periodic orbit.

4. Consider

$$\frac{dx}{dt} = x - y - x(x^2 + 5y^2)$$
$$\frac{dy}{dt} = x + y - y(x^2 + y^2).$$

One can show that the origin is the only critical point (you can use it without verification).

a) Rewrite the system in polar coordinates, using $r'r = xx' + yy'$ and $\theta' = (xy' - yx')/r^2$

b) Find a circle of radius $r_1 > 0$, centered at the origin, such that all trajectories have a radially outward component on it.

c) Find a circle of radius $r_2 > r_1$, centered at the origin, such that all trajectories have a radially inward component on it.

d) Prove that the system has a limit cycle somewhere in the trapping region $r_1 \leq r \leq r_2$.

5. Suppose that a plane system can be written

$$x' = \nabla V(x)$$

for some continuously differentiable single valued scalar function $V$. Such a system is called a gradient system with potential $V$.

What kinds of $\omega$-limit sets can such a system have? (Hint: can there be periodic orbits?)