SF2729 GROUPS AND RINGS
Final Exam
Wednesday, August 15, 2012

Time: 14:00–18:00
Allowed aids: none
Examiners: Wojciech Chachólski and Carel Faber

Present your solutions to the problems in such a way that the arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will be given no points.

For Problem 1, the final score equals at least the number of points obtained from the homeworks of the groups part of the course. If your solution to Problem 1 is worth more points, then this will be your final score.

For Problem 2, the final score equals at least the number of points obtained from the homeworks of the rings part of the course. If your solution to Problem 2 is worth more points, then this will be your final score.

For each problem, the maximum score is 6 points.

The minimum requirements for the various grades are according to the following table:

Grade	**A**	**B**	**C**	**D**	**E**
Total credit	30	27	24	21	18

Problem 1

(6 points). Let G be a finite group whose order is odd (not divisible by 2). Let H be a subgroup of G of index 3. Prove that H is a normal subgroup of G. (Suggestion: consider the action of G on the set of cosets G/H and the induced homomorphism $G \to S_3$).

Problem 2

a. **(4 points)**. Determine a factorization of $g(X) = 4X^3 + 2X^2 - 2X + 6$ into a product of a finite number of irreducibles in each of the following rings:

$$\mathbb{Z}[X], \quad \mathbb{Q}[X], \quad \mathbb{Z}_3[X], \quad \mathbb{Z}_5[X], \quad \mathbb{Z}_7[X].$$

b. **(2 points)**. Determine the prime numbers p for which $g(X)$ factors as a product of 3 linear factors in $\mathbb{Z}_p[X]$.

Please turn over!

Problem 3

Let G be a group. An element g in G is called a **simple commutator** if there are elements a and b in G such that $g = aba^{-1}b^{-1}$. The subgroup of G generated by all the simple commutators in G is denoted by $[G, G]$ and called the commutator subgroup of G.

a. **(1 point)**. Is the conjugation of a simple commutator a simple commutator?

b. **(1 point)**. Show that $[G, G]$ is a normal subgroup in G.

c. **(1 point)**. Prove that $G/[G, G]$ is an abelian group.

d. **(1 point)**. Prove that any cycle of length 3 in the permutation group S_5 is a simple commutator.

e. **(2 points)**. Prove that $[A_5, A_5] = A_5$.

Problem 4

a. **(2 points)**. Determine an irreducible polynomial $f(X)$ of degree 2 in $\mathbb{Z}_3[X]$.

b. **(2 points)**. Determine a generator of the multiplicative group of the finite field

$$\mathbb{Z}_3[X]/\langle f(X)\rangle.$$

c. **(2 points)**. Determine all elements of order 5 in the multiplicative group of the finite field $\mathbb{Z}_{31}$.

Problem 5

Let S_7 be the permutation group of the set $\{1, 2, 3, \ldots, 7\}$.

a. **(1 point)**. Is there a simple commutator in S_7 (see Problem 3) which is an odd permutation?

b. **(3 points)**. Find the number of different elements of order 7 in S_7.

c. **(2 points)**. Find the maximal order of a cyclic subgroup of S_7.

Problem 6

a. **(2 points)**. Denote by K the field $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$. Determine the degree of the field extension $K \supset \mathbb{Q}$.

b. **(2 points)**. Determine an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.

c. **(2 points)**. Prove that $p = 13$ is a prime number such that the reductions modulo p of $X^2 - 2$ and of $X^3 - 2$ both are irreducible in $\mathbb{Z}_p[X]$. Is 13 the smallest such prime?