KTH Teknikvetenskap

SF2729 Groups and Rings
Suggested solutions to midterm exam
Saturday, April 17, 2010

(1) a) Show directly from the axioms that a group G in which $a * a = e$, for all elements a, has to be abelian. **(2)**

b) Find all subgroups of A_4 and write down the subgroup lattice. **(2)**

c) Show that if H and K are finite subgroups of a group G, we have that

$$|HK| = \frac{[H| \cdot |K|}{|H \cap K|},$$

where $HK = \{hk|h \in H, k \in K\}$. **(2)**

SOLUTION

a). If a and b are elements of a group that satisfies $a * a = e$ for all elements a, we get that

$$(a * b) * (a * b) = e$$

and if we multiply this to the left by a and to the right by b, we get

$$a * (a * b) * (a * b) * b = (a * a) * (b * a) * (b * b) = (a * e) * b$$

using the associativity. We now use that $a * a = b * b = e$ and that $a * e = a$ which together with the associativity yields

$$e * (b * a) * e = a * b$$

and hence

$$b * a = a * b,$$

since e is a unit. We conclude that the group has to be abelian if $a * a = e$ for all $a \in G$.

b). A_4 consists of the twelve even permutations in S_4. Each of the elements generate a cyclic subgroup. The elements of order two $(ij)(k\ell)$ generate subgroups of order two and the eight elements of order three generate subgroups of order three. Thus we get one cyclic subgroup of order 1, three of order 2 and four of order 3. By Lagrange's theorem, there could be subgroups of order 1, 2, 3, 4, 6 and 12. We have already found all the subgroups of order 1, 2 and 3, since these have to be cyclic. If there is a subgroup of order 4 it has to have only elements of order 1 and 2, and since there are only four such elements, there is a unique possibility. This is an abelian subgroup, since the elements of order two commute. We are left to find the subgroups of order 6, if there are any. Such a subgroup cannot be cyclic. Hence it must be isomorphic to S_3. However, S_3 contains three elements of order two, and they don't all commute. Since we only have three elements of order two in A_4 and they all commute, there cannot be a subgroup isomorphic to S_3. Hence there are no subgroups of order 6. The only containments between the proper non-trivial subgroups are between the subgroup of order four and the three subgroups of order two.

c). Since every element in HK can be written as hk, where $h \in H$ and $k \in K$, we get that HK is the union of the cosets of K which contains elements of H. This union is a partition of HK since cosets are disjoint. We now have to count the number of such cosets. $hK = h'K$ means that $h^{-1}h' \in K$, but this happens only if $h^{-1}h' \in H \cap K$. Thus the number of cosets in HK is given by $|H|/|H \cap K|$ and since each coset has cardinality $|K|$, we get that

$$|HK| = \frac{|H|}{|H \cap K|} \cdot |K| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

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(2) a) Define what a normal subgroup is and show that there is a well-defined group structure on the set of cosets of a normal subgroup H of a group G. **(2)**

b) Let G be the group of upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

in $\mathrm{Gl}_3(\mathbb{Z}_3)$. Determine the center $Z(G)$ and compute the factor group $G/Z(G)$. **(4)**

SOLUTION

a). A subgroup $H \leq G$ is normal if the left and right cosets agree, which means that $aH = Ha$ for all $a \in G$ or equivalently $aHa^{-1} = H$, for all $a \in H$. We define the group structure on the factor group G/H for a normal subgroup H by

$$aH * bH = abH.$$

We can see that this is well-defined since

$$aHbH = abHH = abH$$

as sets. Furthermore, associativity holds since

$$\begin{aligned}(aH * bH) * cH &= (abH) * cH = (ab)cH \\ &= a(bc)H = aH * bcH = aH * (bH * cH).\end{aligned}$$

The coset eH is a unit since

$$eH * aH = eaH = aH, \qquad \forall aH \in G/H.$$

The inverse of the coset aH is given by the coset $a^{-1}H$ since

$$aH * a^{-1}H = aa^{-1}H = eH, \qquad \forall aH \in G/H.$$

Thus we have a well-defined group structure on the set of cosets, G/H.

b). When we multiply two of the matrices we get

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & b+b'+ac' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix}$$

In order for these matrices to commute, we need $ac' = a'c$. Thus if the firts matrix is in the center, we have $ac' = a'c$ for all choices of a', c'. In particular, with $a' = 1$ and $c' = 0$, we get $c = 0$ and with $a' = 0$ and $c' = 1$ we get $a = 0$. On the other hand, if $a = c = 0$, we always get $ac' = a'c$. Thus the center is given by

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{Z}_3 \right\}.$$

We can compute the cosets as

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} Z(G) = \left\{ \begin{pmatrix} 1 & a & b+b' \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : b' \in \mathbb{Z}_3 \right\}.$$

When we multiply two cosets, we add the values of a and c. Thus we get that the factor group $G/Z(G)$ is isomorhic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

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(3) a) Let H and K be normal[1] subgroups of a group G such that $HK = G$ and $H \cap K = \{e\}$. Show that $G \cong H \times K$ **(2)**

b) The symmetric group S_4 can be presented by the generators $\{s_1, s_2, s_3\}$ and the relations $s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^3 = (s_1s_3)^2 = (s_2s_3)^3 = e$. Use this in order to determine the automorphism group of S_4. **(4)**

SOLUTION

a). We can use the second isomorphism theorem since K is in the normalizer of H and vice versa. We get that $G/K \cong H$ and $G/H \cong K$. Thus we can define a homomorphism

$$\Phi : G \longrightarrow G/K \times G/H \cong H \times K$$

by $\Phi(g) = (gK, gH)$. The kernel is given by g such that $gK = K$ and $gH = H$, which is just $g \in H \cap K$. Thus Φ is injective. Furthermore, we have that if $h \in H$ and $k \in K$, we get that

$$\Phi(hk) = (hkK, hkH) = (hK, hHk) = (hK, Hk) = (hK, kH)$$

which shows that Φ is surjective.

[1]This was unfortunately not mentioned in original version of the exam.

b). The genators must be mapped to elements which satisfies the same relations. Thus they need to be mapped to elements of order 2. We have nine elements of order 2, the six simple transpositions (ij) and the three products of commuting transpositions, $(ij)(k\ell)$. However, the latter are even permutations, and the generators are odd. Hence we are forced to send s_1, s_2 and s_3 to simple transpositions. Once we decide on the image of s_1, there is a unique transposition which commutes with it. Thus we have six choices for the images of s_1 and s_3. Once we decided this, we can choose the image of s_2 to be any of the four remaining transpotitions, since the product of any of these with the two chosen ones are all of order three. Thus we have $6 \cdot 4$ different automorphisms. In order to get the group structure of the automorphism group, we can check that we have 24 different inner automorphisms, since the center of S_4 is trivial. Therefore, the automorphism group is isomorphic to S_4 and can be seen as the group of inner automorphisms of S_4.