SF2729 Groups & Rings

Homework 3: Cyclic groups

1. Let G be a group. The *center* of G is defined as $Z(G) = \{g \in G; gx = xg \text{ for all } x \in G\}$. Show that $Z(G)$ is a subgroup of G and calculate $Z(S_3)$.
2. Let G be a group and H, K be subgroups of G. The *intersection* of H and K is defined as $H \cap K = \{g \in G; g \in H \text{ and } g \in K\}$. It is a subgroup of G. Let $G = \langle g \rangle$ be a cyclic group of order n with generator g. Suppose that d and e divides n. Calculate the order of $\langle g^d \rangle \cap \langle g^e \rangle$.
3. The multiplication in the symmetric group is defined as function composition. In practical calculations however, most people prefer to calculate the product of permutations $\sigma\tau$ as "first σ, then τ" rather than "first τ, then σ". Show that this doesn't matter. More precisely, let $(G, *)$ be a group let $(G, *')$ be the group consisting of the same set of elements as $(G, *)$, but with multiplication defined as $g *' h = h * g$. Then show that $(G, *)$ and $(G, *')$ are isomorphic.
4. **Voluntary exercise.** Euler's totient function $\varphi(n)$ can be defined as the number of generators of a finite cyclic group of order n. Show, using the structure theorem of the subgroups of a finite cyclic group, that

$$\sum_{d|n} \varphi(d) = n$$

where the summation is taken over the divisors d of n.