SF2729 Groups & Rings

Homework 4: Permutations

1. Let $n \geq 2$ be an integer. Let $\sigma \in S_n$ be a permutation. Show that the order of σ is the least common multiple of the lengths of its cycles. For a hint, see the book, page 107.
2. Let $n \geq 2$ be an integer. Let $\alpha, \tau \in S_n$, where α is the k-cycle $(a_1\ a_2\ \cdots\ a_k)$. Show that
$$\tau\alpha\tau^{-1} = (\tau(a_1)\ \tau(a_2)\ \cdots\ \tau(a_k)).$$
3. Use previous exercise to show that if ρ and τ are transpositions, then there is a permutation σ such that $\sigma\rho\sigma^{-1} = \tau$.
4. A (complex) linear character of a group G is a homomorphism from G into the multiplicative group of the complex numbers. That is, a homomorphism $G \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Show that the signature is the only non-trivial linear character of S_n by first showing that if $\chi : S_n \to \mathbb{C}^*$ is linear character, then $\chi(\sigma\tau\sigma^{-1}) = \chi(\tau)$ for all permutations $\sigma, \tau \in S_n$. In particular, using previous exercise, χ is constant on the transpositions. Now use the fact that $1 = \chi(\tau^2) = (\chi(\tau))^2$ if τ is a transposition and that any permutation may be written as a product of transpositions to conclude that χ, if non-trivial, is indeed the signature.