SF2729 Groups & Rings

Homework 6: Action.

1. Let G be a finite group and $H, K \leq G$ be subgroups. Denote by HK the set $\{hk; h \in H, k \in K\}$. Show that
$$|HK| = \frac{|H||K|}{|H \cap K|}.$$
(Hint: Let $X = G/K$ and let H act on X by left multiplication. What is the orbit and the stabilizer of $K \in X$?)
2. Let G be a group and suppose that $N \trianglelefteq G$. Let $g \in N$. Show that the conjugacy class of g in G is a subset of N and that it is a union of conjugacy classes of N.
3. Let G be a group of order $2m$, where m is odd. Show that G has a subgroup of order m, which then must be normal. (Hint: Let G act on itself by left multiplication, and so obtain a homomorphism $G \to S_G$. By an earlier exercise, there is an element, $g \in G$, of order 2. What does the image of g in S_G look like, if you use cycle notation? Show that it is an odd permutation and consider the kernel of the homomorphism $G \to S_G \xrightarrow{\text{sgn}} \{\pm 1\}$.)