Suggested exercises, Lecture 2

Galois theory

1. Show that $x^3 + x^2 + x + 2$ is irreducible in $\mathbf{Q}[X]$ (use the Gauss lemma). Show also that $x^3 + x^2 + x + 2$ is reducible in $\mathbf{R}[X]$.

2. Let $\alpha \in \mathbf{R}$ be a zero of $x^3 + x^2 + x + 2$. Consider the ring $\mathbf{Q}[\alpha] \subset \mathbf{R}$. Show that this ring is a field and express $(\alpha^2 + \alpha + 1)((\alpha^2 + \alpha)$ and $(\alpha - 1)^{-1}$ in the form $a\alpha^2 + b\alpha + c$.

3. Let $K \subset F_1 \subset E$ and $K \subset F_2 \subset E$ be a field extensions. Assume that F_1 and F_2 are finite over K. Recall that $F_1F_2 \subset E$ is the smallest subfield in E containing both F_1 and F_2. Show that F_1F_2 is finite over K and that $[F_1F_2 : K] \leq [F_1 : K][F_2 : K]$. Furthermore, prove that if $[F_1 : K]$ and $[F_2 : K]$ are relatively prime numbers, then $[F_1F_2 : K] = [F_1 : K][F_2 : K]$.