Suggested exercises, Lecture 3&4

Galois theory

1. Let $E = F(\alpha)$ where α is algebraic over F of odd degree (the irreducible polynomial of α has odd degree). Show that $E = F(\alpha^2)$

2. Let $F \subset E$ be a field extensions and α, β in E be algebraic elements over F. Let $f \in F[X]$ be the irreducible polynomial for α and $g \in F[X]$ be the irreducible polynomial for β. Assume that $\deg(f)$ and $\deg(g)$ are relatively prime. Show that g is irreducible in $F(\alpha)[X]$.

3. Show that $\sqrt{2} + \sqrt{3}$ is algebraic over $\mathbf{Q}$ and find its degree.

4. Describe the splitting fields of the following polynomials in $\mathbf{Q}[X]$:

a. $X^2 - 2$
b. $X^2 - 1$
c. $X^3 - 2$
d. $(X^3 - 2)(X^2 - 2)$
e. $X^2 + X + 1$
f. $X^6 + X^3 + 1$
g. $X^5 - 7$

5. Let $\alpha \in \mathbf{R}$ be a root of $X^4 - 5$ (α is real).

a. Show that $\mathbf{Q} \subset \mathbf{Q}(i\alpha^2)$ is normal.
b. Show the inclusion $\mathbf{Q}(i\alpha^2) \subset \mathbf{Q}(\alpha + i\alpha)$ and that it is a normal extension.
c. Show that $\mathbf{Q} \subset \mathbf{Q}(\alpha + i\alpha)$ is not normal.