Suggested exercises, Lecture 5&6

Galois theory

1. Let k be a field and $f = x^l + a_1 x^{l-1} + \cdots + a_l$ be a separable polynomial in $k[X]$. Let $k \subset F$ be a splitting field of f. Assume that all the roots of f form a **subfield** of F. Show that $\mathrm{char}(F) = p > 0$ and $f = X^{p^k} - X$.

2. Let $\mathrm{char}(F) = p > 0$. Show that if $F \subset L$ is a finite field extension such that p does not divide $[L : F]$, then $F \subset L$ is separable.

3. Let $\mathrm{char}(F) = p > 0$. Let $a \in F$ be an element such that $X^p - a$ has no root in F. Show that, for any positive integer n, $X^{p^n} - a$ is irreducible in $F[X]$.

4. Let $\mathrm{char}(F) = p > 0$. Let $F \subset K$ be an algebraic extension. Show that $\alpha \in K$ is separable over F if and only if for any positive integer n, $F(\alpha) = F(\alpha^n)$.

5. Let $\omega = e^{2\pi i/3} \in \mathbf{C}$. Show that $\mathbf{Q}(\omega, \sqrt{5}) = \mathbf{Q}(\omega\sqrt{5})$.

6. Find a primitive element of $\mathbf{Q} \subset \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.