Fourier Analysis

1) Book Stein & Shakarich.

2) Every Tuesday 1-3 pm here

3) Examination.

i) For E-D weekly assignments
(Not all will be assessed)

ii) FOR C-B You need a D on the assignments and write an exam

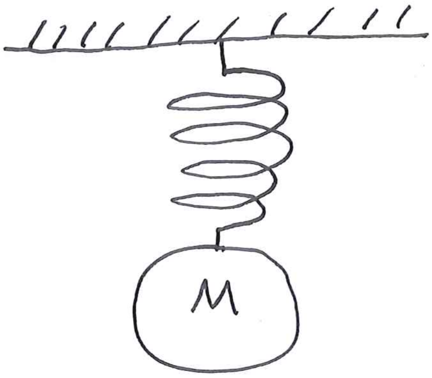
iii) For A you need a B and pass an oral exam.

4) We will follow chapter 1-6 in Stein - Shakarchi's book for the first 10 (or so) lectures.

Then we will do something else (maybe chapter 7-8).

What I really want to do is mathematics

Let M be a weight of mass m hanging in a spring (without any gravity) such



that the positio at time t is $y(t)$.

We assume that the spring exerts a force $-k\,y(t)$ on M and that Newtons law $m y''(t) = F = -k y(t)$ applies. That is $y(t)$ satisfies the differential eq.

$$y'' + \overset{c^2}{\cancel{\tfrac{k}{m}}}\, y = 0 \qquad (1).$$

where $c = \sqrt{\frac{k}{m}}$.

We know that the solution has the form

$$y(t) = a\cos(ct) + b\sin(ct). \quad \text{for some } a, b \in \mathbb{R}.$$

This is extremly simple (first year analysis).

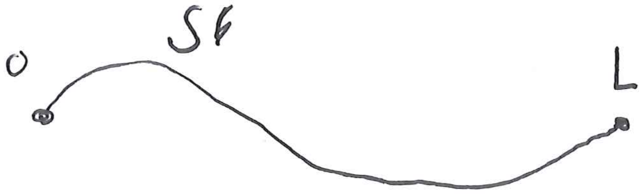
Given some initial data

$$y(t_0) = A \quad \& \quad y'(t_0) = B$$

we can solve $y(t)$.

Example 2. (Vibrating string)

Question? ~~Pde~~ What equation determines the evolution of a vibrating string?



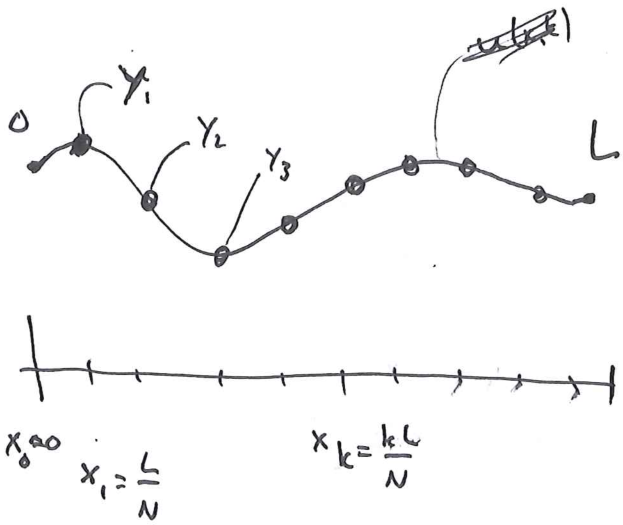
Two things seems to be of importance

1) The mass of the string. M

2) The tension in the string. τ

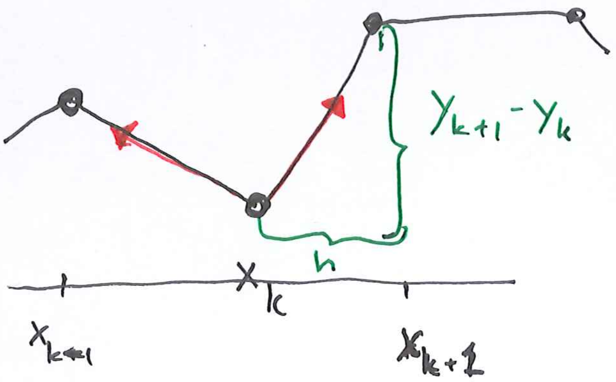
Assume τ to be independent of x and t.

Discretisation: Let us look at the "approximating" problem with a weightless string with N (= large number) of masses $\frac{M}{N}$ equally distributed along the string. That is



There is a mass $\frac{M}{N}$ at each point $\frac{kL}{N}$ along the string.

We will call $\frac{L}{N}=h$.



The force acting on the y direction (x_k, y_k) from the right is

$$T \sin\left(\arctan\left(\frac{y_{k+1}-y_k}{h}\right)\right) =$$

$$= T \frac{\frac{y_{k+1}-y_k}{h}}{\sqrt{1+\left(\frac{y_{k+1}-y_k}{h}\right)^2}} \approx \{\text{Taylor}\}$$

$$\approx T \frac{y_{k+1}-y_k}{h}\left(1 - \frac{1}{2}\left(\frac{y_{k+1}-y_k}{h}\right)^2\right) \approx \left\{\text{Assume that } \frac{y_{k+1}-y_k}{h} \text{ is "small"}\right\} \approx$$

$$\approx T \frac{y_{k+1}-y_k}{h}$$

Similarly, the force acting on (x_k, y_k) from the left point is approximately

$T \frac{y_{k-1}-y_k}{h}$ so the total vertical force acting on x_k is $T \frac{y_{k+1}-2y_k+y_{k-1}}{h}$

So newtons law of acceleration gives

$$\frac{M}{L} h \frac{\partial^2 y_k(t)}{\partial t^2} = T \frac{y_{k+1}-2y_k+y_{k-1}}{h}$$

That is if we write $y_k(t) = y(x_k, t)$

$$\frac{\partial^2 y_k(t,x_k)}{\partial t^2} = \frac{\tau L}{M} \frac{y(t,x_k+h) - 2y(t,x_k) + y(t,x_k-h)}{h^2}$$

If we let $h \to 0$ (That is $N \to \infty$) we arrive at

$$\frac{\partial^2 y(t,x)}{\partial t^2} = \frac{\tau L}{M} \frac{\partial^2 y(t,x)}{\partial x^2}, \quad \Big\} \text{ Equations for a vibrating string}$$

Remarks:

1) Discuss this deduction 5 min.

2) <u>Not mathematics!</u>

i) too sloppy with Taylor expansions and throwing away terms

ii) k is fixed but when $N \to \infty$ we "keep the same point" easy to fix.

iii) we assume that $\dfrac{y(t,x+h) - 2y(t,x) + y(t,x-h)}{h}$ converges

iv) We disregard forces in the x-direction

v) We do not formalize things mathematically but rely on physics!

3) We will not attempt to do this with mathematical stringency but assume that $\frac{\partial^2 u}{\partial t^2} = \lambda^2 \frac{\partial^2 u}{\partial x^2}$ is an important equation given by God.

Change of variables: Let

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad \text{for all} \quad x \in [0,L]$$

and $aX = x$, $bT = t$ and $U(X,T) = u(x,t)$

Then $U(X,T)$ is defined in $[0, \frac{L}{a}]$

and $$\frac{\partial^2 U}{\partial T^2} = \left\{ \begin{aligned} \frac{\partial}{\partial T} &= \frac{\partial t}{\partial T}\frac{\partial}{\partial t} + \frac{\partial x}{\partial T}\frac{\partial}{\partial x} = \\ &= \frac{1}{b}\frac{\partial}{\partial t} \end{aligned} \right\} = \frac{1}{b^2}\frac{\partial^2 u(x,t)}{\partial t^2} =$$

$$\frac{\lambda^2}{b^2}\frac{\partial^2 u(x,t)}{\partial x^2} = \left\{ \begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial T}{\partial x}\frac{\partial}{\partial x} = \\ &= a\frac{\partial}{\partial X} \end{aligned} \right\} = \frac{\lambda^2 a^2}{b^2}\frac{\partial^2 U(X,T)}{\partial X^2}$$

so we can choose $a = \frac{L}{\pi}$ and $b = \frac{\pi}{\lambda L}$

and get a new equation of the same type

$$\frac{\partial^2 U(X,T)}{\partial T^2} = \frac{\partial^2 U(X,T)}{\partial X^2} \qquad \text{in} \quad [0,\pi].$$

It is therefore no loss of generality to assume that $L = \pi$ (or whatever else we fancy)

and $\lambda = 1$.

Important! In the future we will often, and without loss of generality, assume that a function $f(x)$ is defined on a convenient interval

Question: Given some $f(x)$ and $g(x)$ defined on $[0,\pi]$

can we find an $u(x,t)$ s.t.

1. $\dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial^2 u(x,t)}{\partial x^2}$ in $[0,\pi]\times\{t>0\}$

2. $u(x,0) = f(x)$ for $x\in[0,\pi]$

3. $\dfrac{\partial u(x,0)}{\partial t} = g(x)$ for $x\in[0,\pi]$

4. $u(0,t) = u(\pi,t) = 0$ for $t>0$

Any ideas?

An absolutely crazy (and wonderfull) idea:

Assume that $u(x,t) = e(x)\Psi(t)$

If $u(x,t) = e(x)\Psi(t)$ then (1) becomes

$$e(x)\Psi''(t) = \Psi(t)e''(x) \underset{(\Psi\neq 0,\ e\neq 0)}{\Longrightarrow} \frac{\Psi''(t)}{\Psi(t)} = \frac{e''(x)}{e(x)} = -\lambda .$$

$$\Rightarrow \begin{array}{l}\Psi''(t) = -\lambda\Psi(t)\\ e''(x) = -\lambda e(x)\end{array} \Longrightarrow \begin{array}{l}\Psi(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)\\ e(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)\end{array}$$

[Here we assume that $\lambda\geq 0$, a simple calculation shows that if $\lambda = -\mu^2 < 0$ then

$e(x) = Ae^{\mu x} + Be^{-\mu x}$ but using (4) gives

$e(0) = 0 \Rightarrow A = -B$ and

$e(\pi) = A(e^{\mu\pi} - e^{-\mu\pi}) = 0 \Rightarrow A = 0$ so $e = 0$]

Now (4) states that $\varphi(0) = \varphi(\pi) = 0$

That is $A \cdot \cos(0) + B \cdot \sin(0) = A = 0$

and $\underbrace{A\cos(\sqrt{\lambda}\,\pi)}_{=0} + B\sin(\sqrt{\lambda}\,\pi) = 0$

so if $B \neq 0$ then $\sqrt{\lambda} \in \mathbb{Z}$ so $\lambda = n^2$ for some $n \in \{0, 1, 2, \dots\}$.

So we may conclude that

$$u(x,t) = \sum_{n=0}^{\infty} \cancel{B} \sin(nx)\left(A_n \cos(nt) + B_n \sin(nt)\right). \qquad (5)$$

But the equation is linear so we may add terms corresponding to different n and get a new solution.

The solution in (5) satisfies (1) and (4)

But does it satisfy (2) and (3) ?

Condition 2 states that

$$\sum_{n=0}^{\infty} A_n \sin(nx) = f(x) \qquad (6)$$

and 3 that

$$\sum_{n=0}^{\infty} n B_n \sin(nx) = g(x) \qquad (7).$$

Main question: (Fourier series)

Given a function $f(x)$ on $[0,\pi]$ can we write

$$f(x) = \sum_{n=0}^{\infty} A_n \sin(nx) + B_n \cos(nx) \qquad ?$$

So the absolutely crazy idea leads to ~~whether~~ we can write a function $f(x)$ as an infinite series of sin & cos.

This is an absolutely pregnant idea that has influenced very very much mathematics over the centuries

Mathematics.

If $\quad f(x) = \sum_{n=0}^{\infty} A_n \sin(nx) + B_n \cos(nx) \qquad$ for $x \in [0, \pi]$

then, given that the integrals exsist,

$$\int_0^{\pi} f(x) \sin(kx)\, dx = \int_0^{\pi} \left(\sum_{n=0}^{\infty} A_n \sin(nx) \sin(kx) + \sum_{n=0}^{\infty} B_n \underbrace{\cos(nx)\sin(kx)}_{=0} \right) dx =$$

$$= A_k \cdot \pi/2 \qquad \text{~~and similarly~~} \Rightarrow A_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)\, dx$$

~~Since~~

$$B_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx)\, dx$$

Questions: C

1) For what functions (if any) does

$$f(x) = \lim_{N \to \infty} \sum_{n=0}^{N} \left(A_n \cos(nx) + B_n \sin(nx) \right)$$

and in what sense?

Where

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

Fourier analysis.

- Q: ~~How far~~ what can be said of the set where $\sum A_n \sin + B_n \cos$ is discent $\xrightarrow{\text{Cantor}}$ Transinfinite sets

- View $\sin(nx)$ & $\cos(nx)$ as a basis for a vector space $\xrightarrow{\text{Hilbert}}$ Functional Analysis

- General Convergence questions $\Rightarrow$ Analysis Riemann integral

- Ψdo differential operators

- Analytic number theory