The Fourier Transform on $\mathbb{R}$

(continued)

In our last lecture we introduced the space of Schwartz functions on the real line, denoted by $\mathcal{S}(\mathbb{R})$.
Then we defined the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R})$ by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx,$$

and showed that

(a) $\widehat{f'(x)}(\xi) = 2\pi i \xi \hat{f}(\xi)$

(b) $\widehat{-2\pi i x f(x)}(\xi) = \frac{d}{d\xi} \hat{f}(\xi)$.

And proved the following

Theorem: If $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$.

Question: Is it possible to recover a function from its Fourier transform?
In other words: given $F \in \mathcal{S}(\mathbb{R})$, can we find $f \in \mathcal{S}(\mathbb{R})$ such that $\hat{f}(\xi) = F(\xi)$?

The answer is yes, and it can be expressed in a formula (called inversion formula).
In order to prove it, we study

The Gaussians as good kernels.

Let us consider the Gaussian function,

$f(x) = e^{-ax^2}$, with $a = \pi$.

We have seen that $f \in \mathcal{S}(\mathbb{R})$, we go on to study its further properties.

First we show that

$$(1) \qquad \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$$

This can be proven in a very nice way by noticing that

$$\left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} e^{-\pi (x^2 + y^2)} dx\, dy =$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-\pi r^2} r \, dr \, d\theta = \int_0^{\infty} 2\pi r \, e^{-\pi r^2} dr \, d\theta =$$

$$= \left[e^{-\pi r^2} \right]_1^{\infty} = 1$$

This is the normalization property, explaining our choice of $a = \pi$.

From (1) and properties of the Fourier transform, we can prove the fundamental property of the Gaussian:

Theorem 1: If $f(x)=e^{-\pi x^2}$, then $\hat{f}(\xi)=f(\xi)$.

Proof: Define

$$F(\xi)=\hat{f}(\xi)=\int_{-\infty}^{\infty} f(x)\,e^{-2\pi i x\xi}\,dx,$$

and observe that $f'(x)=-2\pi i x f(x)$.

Then

$$F'(\xi)\overset{(b)}{=}\widehat{(-2\pi i x f(x))}(\xi)=\int_{-\infty}^{\infty} f(x)(-2\pi i x)e^{-2\pi i x\xi}\,dx=$$

$$=i\int_{-\infty}^{\infty} f'(x)\,e^{-2\pi i x\xi}\,dx\overset{(a)}{=}i(2\pi i\xi)\hat{f}(\xi)=$$

$$=-2\pi\xi F(\xi)$$

So we get

$$F'(\xi)=-2\pi\xi F(\xi),$$

Define $G(\xi)=F(\xi)\,e^{\pi\xi^2}$, then

$$G'(\xi)=(F'(\xi)+2\pi\xi F(\xi))e^{\pi\xi^2}=0,$$

hence $G(\xi)\equiv\text{const}$, we know that

$G(0)=F(0)\overset{(1)}{=}1$. So $G(\xi)\equiv 1$, and

$$\hat{f}(\xi)=F(\xi)=e^{-\pi\xi^2}=f(\xi).$$

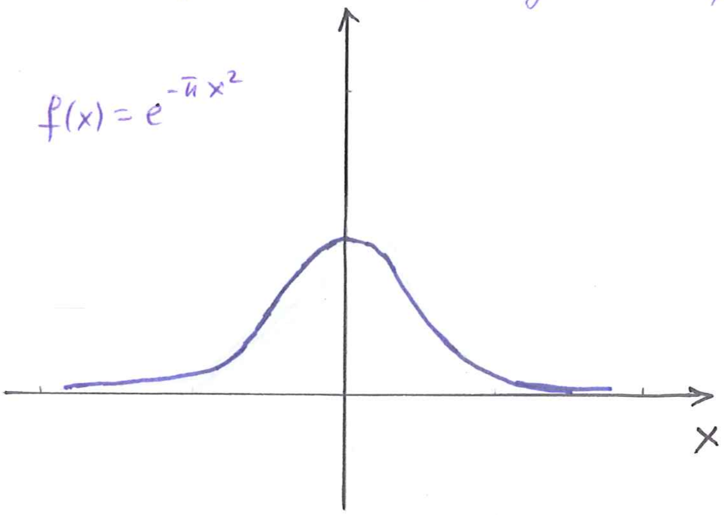
Corollary: If $\delta>0$ and $K_\delta(x)=\delta^{-1/2}\,e^{-\pi x^2/\delta}$, then $\widehat{K_\delta}(\xi)=e^{-\pi\delta\xi^2}$

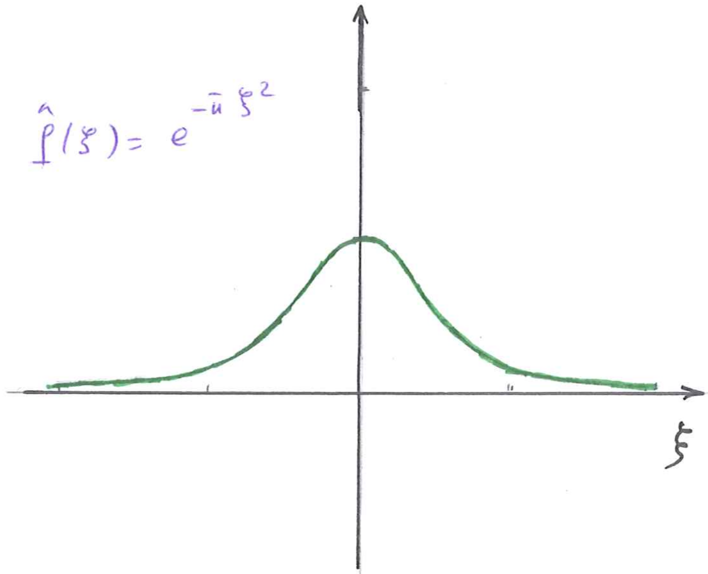
The proof follows from the scaling properties of the Fourier transform and the theorem.

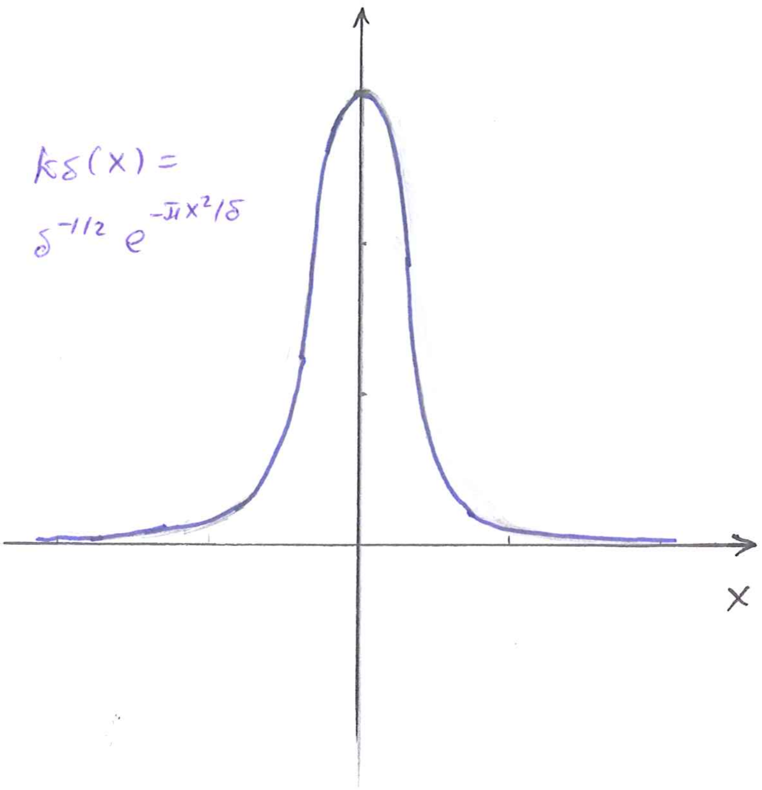
(exercise)

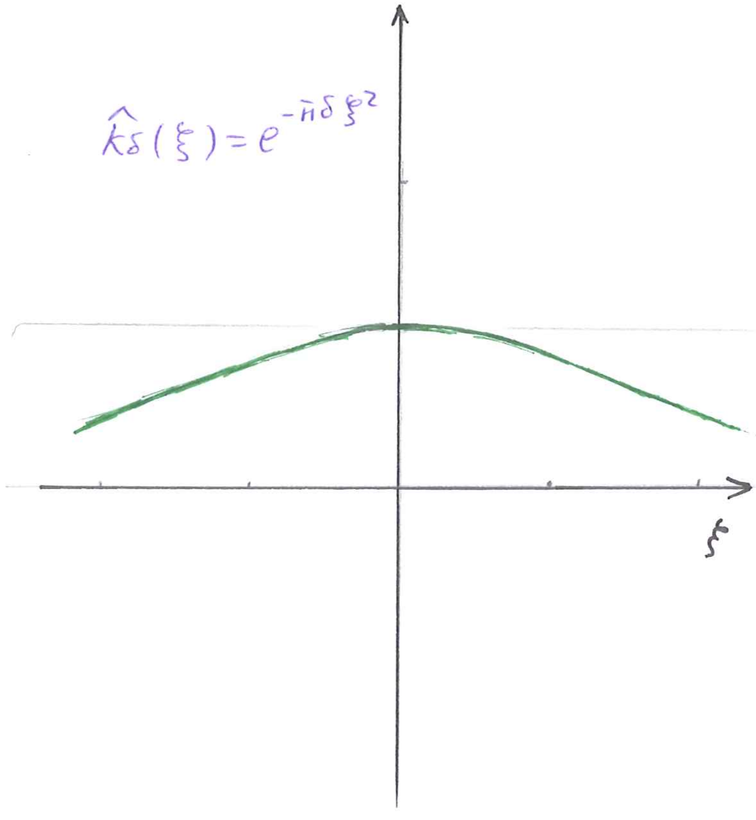
We see that k_δ and $\hat{k}_\delta$ cannot both be concentrated at the origin. This is an example of a general phenomenon, called the <u>Heisenberg uncertainty principle</u>, which will be discussed later on.

For now we only mention that as $\delta \to 0$ k_δ peaks at the origin, while its Fourier transform $\hat{k}_\delta$ gets flatter.









Theorem 2: The collection $\{k_\delta\}_{\delta>0}$ is a family of good kernels on the real line:

Proof: We have that

$$k_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta}, \quad x \in \mathbb{R}, \quad \delta > 0$$

we need to verify that

(i) $\int_{-\infty}^{\infty} k_\delta(x)dx = 1$

(ii) $\int_{-\infty}^{\infty} |k_\delta(x)|\, dx \leq M$

(iii) For every $\eta > 0$, we have

$$\int_{|x|>\eta} |k_\delta(x)|dx \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

Proof: (i) $\int_{-\infty}^{\infty} k_\delta(x)dx = \hat{k}_\delta(0) = 1$

(ii) holds since, k_δ is nonnegative

(iii) $\forall\, \eta > 0$ fixed

$$\int_{|x|>\eta} |k_\delta(x)|dx = \int_{|y|>\eta/\delta^{1/2}} e^{-\pi y^2} dy \to 0 \quad \text{as } \delta \to 0.$$

For a pair of functions $f, g \in S(\mathbb{R})$, the <u>convolution</u> is defined by

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)\,dt$$

<u>Corollary:</u> If $f \in S(\mathbb{R})$, then

$$(f * K_\delta)(x) \longrightarrow f(x) \qquad (*)$$

uniformly in x as $\delta \to 0$.

<u>Proof:</u> First we show that f is uniformly continuous in $\mathbb{R}$: given $\varepsilon > 0$ $\exists R > 0$ s.t. $|f(x)| < \frac{\varepsilon}{4}$, $\forall |x| \geq R$, and f is uniformly continuous in $[-R, R]$, hence $\exists \eta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \eta$.

From (i), we have that

$$(f * K_\delta)(x) - f(x) = \int_{-\infty}^{\infty} K_\delta(t)[f(x-t) - f(x)]\,dt,$$

$\Rightarrow$

$$|(f * K_\delta)(x) - f(x)| \leq \int_{|t|>\eta} K_\delta(t)|f(x-t) - f(x)|\,dt +$$

$$+ \int_{|t|\leq\eta} K_\delta(t)\,|f(x-t) - f(x)|\,dt < 2\sup_{x\in\mathbb{R}}|f(x)| \int_{|t|>\eta} K_\delta(t)\,dt$$

$$+ \varepsilon \int_{|t|\leq\eta} K_\delta(t)\,dt \leq \varepsilon + \varepsilon$$

since $\int_{|t|>\eta} K_\delta(t)\,dt \to 0$ as $\delta \to 0$ and

$$\int_{|t|\leq\eta} K_\delta(t)\,dt \leq 1$$

The Fourier Inversion.

The interchange of order of integration for double integrals: Suppose $F(x,y)$ is a continuous function in $\mathbb{R}^2$, and assume

$$|F(x,y)| \leq \frac{A}{(1+x^2)(1+y^2)}$$

Then $F_1(x) := \int_{-\infty}^{\infty} F(x,y)\,dy$ is well defined, and F_1 is a continuous function of moderate decrease. Similarly, define $F_2(y) = \int_{-\infty}^{\infty} F(x,y)\,dx$, then $F_2 \in \mathcal{M}(\mathbb{R})$. So both F_1 and F_2 are integrable in $\mathbb{R}$, moreover

$$\int_{-\infty}^{\infty} F_1(x)\,dx = \int_{-\infty}^{\infty} F_2(y)\,dy \qquad (**).$$

in order to prove it, let us consider

$$\int_{-M}^{M} F_1(x)\,dx = \int_{-M}^{M}\left[\int_{-N}^{N} F(x,y)\,dy\right]dx + \int_{-M}^{M}\left[\int_{|y|\geq N} F(x,y)\,dy\right]dx$$

then. $$\int_{-M}^{M}\left[\int_{|y|\geq N} |F(x,y)|\,dy\right]dx \leq A\int_{-M}^{M} \frac{1}{1+x^2}\left[\int_{|y|\geq N} \frac{1}{y^2}\,dy\right]dx =$$

$$= \frac{2A}{N}\int_{-M}^{M} \frac{dx}{1+x^2},$$ similarly

$$\int_{-N}^{N}\left[\int_{|y|\geq M} |F(x,y)|\,dx\right]dy \leq \frac{2A}{M}\int_{-N}^{N}\frac{dy}{1+y^2}$$

next we estimate

$$\left|\int_{-M}^{M} F_1(x)dx - \int_{-N}^{N} F_2(y)dy\right| \le \left|\int_{-M}^{M} F_1(x)dx - \int_{-M}^{M}\int_{-N}^{N} F(x,y)dxdy\right|$$

$$+ \left|\int_{-N}^{N} F_2(y)dy - \int_{-N}^{N}\int_{-M}^{M} F(x,y)\,dxdy\right| \le$$

$$\le \frac{2A}{N}\int_{-M}^{M}\frac{dx}{1+x^2} + \frac{2A}{M}\int_{-N}^{N}\frac{dy}{1+y^2} \longrightarrow 0 \quad \text{as } M,N\to\infty$$

hence we get $(**)$.

Let us just mention that ~~we~~ above we used the property of changing the order of integration over rectangles in $\mathbb{R}^2$. ◊

<u>Proposition:</u> If $f,g \in S(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty}\hat{f}(y)g(y)dy.$$

<u>Proof.</u> Consider the function

$$F(x,y) = f(x)g(y)e^{-2\pi i xy},$$

then we see that

$F_1(x) = f(x)\hat{g}(x)$ and $F_2(y) = \hat{f}(y)g(y)$,

and the identity in the proposition is the same as $(**)$.

Now we are ready to prove the Fourier inversion theorem:

<u>Theorem:</u> If $f \in S(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi.$$

<u>Proof:</u> First we see that

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi) d\xi,$$

let $G_\delta(x) = e^{-\pi\delta x^2}$, then $\widehat{G_\delta}(\xi) = K_\delta(\xi)$, by the proposition

$$\int_{-\infty}^{\infty} f(x) K_\delta(x) dx = \int_{-\infty}^{\infty} \hat{f}(\xi) G_\delta(\xi) d\xi$$

letting $\delta \to 0$, we get

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi) d\xi.$$

Now take $\forall\, x \in \mathbb{R}$, let $F(y) = f(y+x)$, then

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(\xi) d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi.$$

So $$f(x) = \widehat{(\hat{f}(\xi))}(-x).$$

We define the Fourier transform as an operator $\mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R})$ by

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

and define another operator

$$\mathcal{F}^* : S(\mathbb{R}) \longrightarrow S(\mathbb{R}) \text{ by}$$

$$\mathcal{F}^*(g)(x) = \int_{-\infty}^{\infty} g(\xi) e^{2\pi i x \xi} d\xi .$$

The inversion theorem implies that $\mathcal{F}^* \circ \mathcal{F} = I$ on $S(\mathbb{R})$, and we have that $\mathcal{F} f(y) = \mathcal{F}^* f(-y) \Rightarrow \mathcal{F} \circ \mathcal{F}^* = I$, so $\mathcal{F}^*$ is the inverse of the Fourier transform and

<u>Corollary</u>: The Fourier transform is a bijective mapping on the Schwartz space.